ANALOG REDUCIBILITY

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Abstract. In this paper we introduce and characterize two 'analog reducibility' notions for $[0,1]$-valued oracles on ω obtained by applying the syntactic characterizations of Turing and enumeration reducibility in terms of (positive) relatively Σ_1 and Π_1 formulas to formulas in continuous logic [5]. The resulting analog and analog enumeration degree structures, $\mathscr{D}_a$ and $\mathscr{D}_{ae}$, naturally extend $\mathscr{D}_T$ and $\mathscr{D}_e$ in a compatible way. To show that these extensions are proper we prove that a sufficiently generic total $[0,1]$-valued oracle does not 'analog enumerate' any non-c.e. discrete set and that a sufficiently generic positive $[0,1]$-valued oracle neither 'analog enumerates' a non-c.e. discrete set nor 'analog computes' a non-trivial total $[0,1]$-valued oracle. We also provide a characterization of the continuous degrees among $\mathscr{D}_{ae}$ as precisely $\mathscr{D}_e \cap \mathscr{D}_a$. Finally we characterize a generalization of r.i.c.e. relations to metric structures via Σ_1 formulas in the 'hereditarily compact superstructure,' which was the original motivation for the concepts in this paper.

Keywords. Continuous Logic, Computable from a Structure, Continuous Degrees, Enumeration Degrees, Superstructures

Introduction

It is well known that Turing reducibility has a purely syntactic characterization. Namely, for sets $X, Y \subseteq \omega$, $X \leq_T Y$ if and only if X is defined by both a $\Sigma_1(Y)$ formula and a $\Pi_1(Y)$ formula [3, Thm. 2.7]. Similarly, enumeration reducibility can be characterized in terms of $\Sigma_1(Y)$ formulas that are positive in Y. Certain basic properties of these reducibility notions follow immediately from syntactic proofs, such as, for instance, transitivity: If $X \leq_T Y \leq_T Z$, then $X \leq_T Z$, because we can take the $\Sigma_1(Y)$ definition of X and turn it into a $\Sigma_1(Z)$ definition of X by replacing positive instances of Y with the $\Sigma_1(Z)$ definition of Y and negative instances of Y with the $\Pi_1(Z)$ definition of Y, and a similar argument gives a $\Pi_1(Z)$ definition of X.

This definition extends directly to the particular notion of computability in a structure, wherein, given a structure $\mathfrak{M}$, we form the 'hereditarily finite superstructure' of $\mathfrak{M}$, $\mathrm{HF}(\mathfrak{M})$, consisting of all hereditarily finite sets built using the elements of $\mathfrak{M}$ as atoms. Σ_1 formulas on this are equivalent to relatively intrinsically computably enumerable (r.i.c.e.) relations on $\mathfrak{M}$.

In this paper we will exploit these syntactic characterizations to generalize these concepts to $[0,1]$-valued oracles on the natural numbers and metric structures in general.

Generalization to Metric Structures. While there is an 'obvious' notion of r.i.c.e. relations on a separable metric structure—namely, relations that are computably enumerable (c.e.) in any presentation of a countable dense sub-structure—this notion does a certain amount of violence to the metrical structure of the metric structure in question, not the least of which is the issue of the choice of countable dense sub-structure. That said, we will show in Proposition 6.9 that it is possible to characterize this notion in terms of the machinery of this paper. The definition of $\mathrm{HF}(\mathfrak{M})$ has an extremely natural generalization to metric structures, which to our knowledge has never been written down before. In Definition 6.1 we present the 'hereditarily compact superstructure,' $\mathrm{HK}(\mathfrak{M})$, which is the metric completion of $\mathrm{HF}(\mathfrak{M})$ under the appropriate 'recursive Hausdorff metric':

$$d^{\mathrm{HF}(\mathfrak{M})}(A,B) = d_H^{\mathfrak{M}}(A \cap \mathfrak{M}, B \cap \mathfrak{M}) \vee d_H^{\mathrm{HF}(\mathfrak{M})}(A \setminus \mathfrak{M}, B \setminus \mathfrak{M}),$$

where A and B are sets in $\mathrm{HF}(\mathfrak{M})$ and d_H^X is the Hausdorff metric on sets in X. This is well defined by the well-foundedness of sets in $\mathrm{HF}(\mathfrak{M})$. This can then be naturally encoded as a metric structure with a binary predicate $E(x,y) = \inf_{z \in y} d(x,z)$, chosen precisely so that relative quantification is well defined.

$\mathrm{HK}(\mathfrak{M})$ can also be built up cumulatively in that it is equal to $\bigcup_{n<\omega} \mathrm{HK}_n(\mathfrak{M})$, where $\mathrm{HK}_0(\mathfrak{M}) = \mathfrak{M}$ and $\mathrm{HK}(\mathfrak{M})_{n+1} = \mathcal{P}_K(\mathrm{HK}_n(\mathfrak{M})) \sqcup \mathfrak{M}$, where $\mathcal{P}_K(X)$ is the collection of all compact subsets of X under the Hausdorff metric. In particular, elements of $\mathrm{HK}(\mathfrak{M})$ have well defined foundational ranks. Furthermore, in the case that $\mathfrak{M}$ is a discrete structure, $\mathrm{HK}(\mathfrak{M})$ is interdefinable with $\mathrm{HF}(\mathfrak{M})$.

The definition of Σ_1 formulas makes sense in continuous logic. This allows us to give a generalization of r.i.c.e. relations to metric structures (specifically, Σ_1 definability in $\mathrm{HK}(\mathfrak{M})$, see Definitions 6.1 and 6.3). At the end of this paper, we will give a characterization of these relations analogous to the characterization of such relations given in [2] and [6].

Analog Degrees. Despite the original motivation of this paper being Σ_1 definability on $\mathrm{HK}(\mathfrak{M})$, the majority of this paper will concern itself with an unavoidable intermediate generalization, namely a kind of 'metric computability' on $\mathrm{HK}(\varnothing)$, or equivalently on ω, concerning itself with 'fuzzy oracles' of the form $P : \omega \to [0,1]$ and relative (continuous) Σ_1 and Π_1 definability of such oracles on structures of the form $(\omega, 0, 1, +, \cdot, P)$. This is given precisely in Definition 1.8.

We call our continuous generalization of Turing reducibility 'analog reducibility,' written $P \leq_a Q$ (Definition 1.9). The name is somewhat inspired by vague intuition regarding continuous behavior of real world electrical circuits, but it is mostly due to the fact that 'continuous degree' is already taken [8]. Since the notion of a formula being positive in a given atomic predicate also makes sense, we also have a continuous generalization of enumeration reducibility, 'analog enumeration reducibility,' written $P \leq_{ae} Q$. These notions reduce to their discrete counterparts on $\{0,1\}$-valued or discrete oracles, and, moreover, the natural inclusion diagram commutes, so the degree structures $\mathscr{D}_T$, $\mathscr{D}_e$, and $\mathscr{D}_a$ all sit as sub-semi-lattices of $\mathscr{D}_{ae}$ (Proposition 2.2). On the other hand, while $\mathscr{D}_a$ and $\mathscr{D}_{ae}$ are motivated as generalizations of $\mathscr{D}_T$ and $\mathscr{D}_e$, it is more accurate to say that they are generalizations of the ω-join semi-lattices of countable Turing ideals and countable enumeration ideals, respectively, as these are also included in a natural way as ω-join semi-lattices.

We will give more traditional computability theoretic characterizations of $\leq_a$ and $\leq_{ae}$. They are equivalent to a sort of 'computability up to uniform error, non-uniformly in scale,' i.e. $P \leq_a Q$ if and only if for every $\varepsilon > 0$ there is a $\delta > 0$ and an index $e < \omega$ such that for any $Q' : \omega \to \mathbb{Q}$ such that $\|Q - Q'\| < \delta$, $\left\| P - \Phi_e^{Q'} \right\| < \varepsilon$. A similar statement is true of $\leq_{ae}$. With a genericity argument we will also show that $P \leq_a Q$ if and only if every $\varepsilon > 0$ there is a $\delta > 0$ such that for any Q' with $\|Q - Q'\| < \delta$ there is a P' with $\|P - P'\| < \varepsilon$ and $P' \leq_T Q'$, with a similar statement for $\leq_{ae}$. This is the content of Theorem 1.18.

The existence of an ω-join on the induced degree structure and the non-uniformity in scale in the above characterization is connected to the liberalness of the definition of continuous formulas, which are closed under uniformly convergent limits. While it could be argued that the more natural notion would be reducibility in terms of 'computable' formulas or a reducibility defined in terms of the equivalent condition requiring a uniformly computable sequence of indices, Corollary 1.20 makes perhaps the strongest case that the presently studied concepts are natural. Corollary 1.20 states that $P \leq_a Q$ if and only if every Turing degree which computes arbitrarily good uniform approximations of Q also computes arbitrarily good uniform approximations of P, as well as a similar fact for $\leq_{ae}$. Compare this to the fact that $X \leq_e Y$ if and only if every Turing degree that computes an enumeration of Y also computes an enumeration of X. We will discuss more restrictive forms of $\leq_a$ and $\leq_{ae}$ in Section 5.

Connection to the Continuous Degrees. There is an existing notion of continuous degrees, introduced by Miller [8]. These degrees are induced by representation reducibility, written $\leq_r$. Although they were not originally defined this way, they can be seen as a degree structure on elements of the Hilbert cube $[0,1]^\omega$. This is of course the same object as the collection of functions from ω to $[0,1]$, but the induced degree structures are at first glance seemingly completely unrelated.

In line with the continuous logic mantra 'anything compact is trivial,' if $\lim_{n\to\infty} P(n) - Q(n) = 0$, then $P \equiv_a Q$ and $P \equiv_{ae} Q$ (where we are thinking of a sequence of reals limiting to 0 as a 'compact' amount of information). This corresponds to the fact that any finite modification of an oracle does not change its Turing or enumeration degree. This is radically different from the behavior of the continuous degrees, in which any two degrees have representations which are asymptotically equivalent (in particular, every continuous degree has a representation P which satisfies $\lim_{n\to\infty} P(n) = 0$). A related consequence of this is that every a or ae degree has a representative whose entries are all rational, whereas a continuous degree has a rational representative if and only if it is actually a Turing degree.

The analog and analog enumeration degree structures are more closely related to $[0,1]^\omega$ with its topology as a subspace of ℓ^∞ than with its topology as the Hilbert cube (although this topology matters too, of course). For example, the set of oracles a- or ae-reducible to some oracle P is always closed in the ℓ^∞ metric, and, while $\leq_a$ and $\leq_{ae}$ do not have the countable predecessor property, they do have the 'separable predecessor property' relative to the ℓ^∞ metric, as in the set of predecessors is always metrically separable. The situation is reminiscent of the topometric structure of type spaces in continuous logic [4], in which there is a natural compact topology and a natural non-compact metric which are compatible in some ways but distinct.

The continuous degrees, $\mathscr{D}_r$, sit between the Turing and enumeration degrees, so it is natural to wonder how they relate to $\mathscr{D}_a$ and $\mathscr{D}_{ae}$. We will show in Proposition 3.9 that the relationship is very tight: as a subset of $\mathscr{D}_{ae}$, $\mathscr{D}_r$ is precisely $\mathscr{D}_e \cap \mathscr{D}_a$. One direction is shown by a direct encoding of continuous degrees as analog degrees (Definition 3.2 and Proposition 3.3). The other direction is almost immediate from Theorem 1.3 in [1], once viewed in the right way.

Generic Analog Degrees. Finally, we will resolve the obvious question of whether or not $\mathscr{D}_a$ has any members that are not countable Turing ideals and whether or not $\mathscr{D}_{ae}$ has any members that are not countable enumeration ideals. Analogously to how a sufficiently generic enumeration oracle has non-trivial degree but also does not compute any non-computable total information, we will show in Proposition 4.3 that a sufficiently generic analog total oracle has non-trivial a degree but does not enumerate any non-c.e. discrete information, and that any sufficiently generic analog enumeration oracle has non-trivial ae degree but neither enumerates any non-c.e. discrete information nor computes any analog total oracle whose degree is not $\mathbf{0}_a$ (Proposition 4.1), resolving both questions in the positive.

1. Analog Reducibility

The following definition is not different in any important way from the corresponding (unnumbered) definition in Section 2 of [5].

Definition 1.1. A **metric signature** $\mathcal{L}$ is a collection of constant symbols $\mathcal{C}$, a collection of function symbols $\mathcal{F}$, and a collection of predicate symbols $\mathcal{P}$, together with

- arities, $a(f)$ and $a(P)$, and
- moduli of uniform continuity, α_f and α_P,

for each $f \in \mathcal{F}$ and $P \in \mathcal{P}$, and with

- **syntactic ranges**, $I(P)$, a compact interval,

for each $P \in \mathcal{P}$. The special predicate symbol d satisfies $a(d) = 2$, $\alpha_d(x) = 2x$, and $I(d) = [0, r]$ for some $r > 0$, the **syntactic diameter** of $\mathcal{L}$.

Terms and free variables for terms are defined in the typical way.

Remark 1.2. In this paper we are considering 1 to be true and 0 to be false (as opposed to the convention common in continuous logic of treating 0 as true), for the sake of compatibility with standard conventions in computability theory.

Definition 1.3. **Formulas** are defined inductively simultaneously with **free variables**, written $\mathrm{fv}(\varphi)$, and **syntactic range**, written $\mathrm{I}(\varphi)$.

(i) If P is an n-ary predicate symbol and $\bar{t}$ is a sequence of n terms, then $P\bar{t}$ is a formula. $\mathrm{fv}(P\bar{t})$ is the union of the collection of free variables in the terms t. $I(P\bar{t}) = I(P)$, the syntactic range of the predicate symbol P.

(ii) If $\bar{\varphi}$ is a sequence of formulas of length $n \leq \omega$ and $F : \mathbb{R}^n \to \mathbb{R}$ is a partial function which is defined and continuous on the set $\prod_{i<n} I(\varphi_i)$, then $F\bar{\varphi}$ is a formula. $\mathrm{fv}(F\bar{\varphi})$ is the union of the free variables in the formulas in

$\bar{\varphi}$, and $I(F\bar{\varphi})$ is the image of the set $\prod_{i<n} I(\varphi_i)$ under F (note that this is always a compact interval).

(iii) If φ is a formula, $\{x_i\}_{i<n}$ is a sequence of distinct variables for some $n \leq \omega$, then

$$\psi = \sup_{x_0} \sup_{x_1} \sup_{x_2} \dots \varphi$$

and

$$\chi = \inf_{x_0} \inf_{x_1} \inf_{x_2} \dots \varphi$$

are formulas. $\mathrm{fv}(\psi) = \mathrm{fv}(\chi) = \mathrm{fv}(\varphi) \setminus \{x_i\}_{i<n}$, and $I(\psi) = I(\chi) = I(\varphi)$.

A **sentence** is a formula φ such that $\mathrm{fv}(\varphi) = \varnothing$. An **atomic formula** is a formula of the form $P\bar{t}$. A **quantifier free formula** is a formula formed without the use of rule (iii). A **restricted formula** is a formula formed only using the connectives 1 (0-ary), $x+y$, $x \wedge y$, $x \vee y$, and $r \cdot x$ for $r \in \mathbb{Q}$ and finitary quantification. (Where $x \wedge y$ and $x \vee y$ are the minimum of x and y and the maximum of x and y, respectively.)

Remark 1.4. Definition 1.3 is broader than the corresponding Definition 3.1 in [5], in that we allow infinitary (continuous) connectives and infinite (non-alternating) strings of quantifiers. Our definition does not add any real expressive power in that everything we refer to as a formula is equivalent to a definable predicate in the sense of [5]. See the discussion after Remark 9.2 in [5]. Also it should be noted that our collection of restricted connectives is slightly larger than those in [5] in that we allow arbitrary rational scaling, which has the advantage of giving a clean normal form (see Definition 1.10).

Remark 1.5. The permissiveness of (ii) in Definition 1.3 is largely for the sake of three particular connectives, namely $\bar{\varphi} \mapsto \sum_{i<\omega} \varphi_i$, $\bar{\varphi} \mapsto \sup_{i<\omega} \varphi_i$, and $\bar{\varphi} \mapsto \inf_{i<\omega} \varphi_i$, which we would like to be able to use freely without equivocating. No essential expressiveness is gained by this permissiveness, since a partial connective F that is continuous on $\prod_{i<n} I(\varphi_i)$ is equal to some continuous total connective G on that set.

We will need a notion of positivity of a formula in a particular predicate symbol.

Definition 1.6.

(i) Let $F : \mathbb{R}^n \to \mathbb{R}$ be a partial function for some $n \leq \omega$ that is defined and continuous on some $\prod_{i<n} I_i$ for some sequence of compact intervals I_i. Let J be some set of indices $< n$. We say that $\boldsymbol{F}$ **is positive for the inputs** $\boldsymbol{J}$ **on** $\boldsymbol{\prod_{i<n} I_i}$ if for any two n-tuples $\bar{x}, \bar{y} \in \prod_{i<n} I_i$ with $x_i = y_i$ for all $i \notin J$ and $x_i \leq y_i$ for all $i \in J$, $F(\bar{x}) \leq F(\bar{y})$.

(ii) Any atomic formula is **positive for** $\boldsymbol{P}$.

If $\bar{\varphi}$ is a sequence of formulas that are positive for P of length $n \leq \omega$, and J is the set of indices $j < n$ such that φ_j contains the predicate P, then if $F : \mathbb{R}^n \to \mathbb{R}$ is positive for the inputs J on $\prod_{i<n} I(\varphi_i)$, then $F\bar{\varphi}$ is positive for P.

If φ is positive for P, then $\sup_x \varphi$ and $\inf_x \varphi$ are positive for P.

Clearly if φ is a formula that is positive for P and $\mathfrak{M}$ and $\mathfrak{N}$ are structures which are identical except for their interpretations of P, which satisfy $P^{\mathfrak{M}} \leq P^{\mathfrak{N}}$, then we have that $\varphi^{\mathfrak{M}} \leq \varphi^{\mathfrak{N}}$.

Since we will be dealing with discrete structures treated as continuous structures or metric structures that are discrete in some sub-signature, it will be useful to have a systematic way of converting discrete formulas into continuous formulas. We will always take discrete predicates P to have $I(P) = [0, 1]$.

The following definition is non-standard, but is an adaptation to our context of the Iverson bracket, which is defined so that $[\varphi] \in \{0, 1\}$ and $[\varphi] = 1$ if and only if φ is true.

Definition 1.7. We define the **syntactic Iverson bracket** of a discrete formula inductively. Let P be a discrete predicate symbol, $\bar{t}$ a tuple of terms, and φ and ψ be discrete formulas.

- $[P\bar{t}] = P\bar{t}$
- $[\varphi \wedge \psi] = [\varphi] \wedge [\psi]$
- $[\varphi \vee \psi] = [\varphi] \vee [\psi]$
- $[\neg\varphi] = 1 - [\varphi]$
- $[\exists x\varphi] = \sup_x[\varphi]$
- $[\forall x\varphi] = \inf_x[\varphi]$

Of course by construction we have that $\mathfrak{M} \models [\varphi] = 1$ if and only if $\mathfrak{M} \models \varphi$ and that $[\varphi]$ is always $\{0, 1\}$-valued. Note that a syntactic Iverson bracket is always a restricted formula.

Definition 1.8. In a structure of the form $(\omega, 0, 1, +, \cdot, P)$, with $P : \omega \to [0, 1]$ an arbitrary function, given a formula φ with $I(\varphi) = [a, b]$, a variable x, and a term t,

- $\sup_{x<t} \varphi$ represents the formula $\sup_x(b + (a - b)[x < t]) \wedge \varphi$, and
- $\inf_{x<t} \varphi$ represents the formula $\inf_x(a + (b - a)[x < t]) \vee \varphi$.

These are called **bound quantifiers**. A $\boldsymbol{\Delta_0(P)}$ **formula** is a formula formed from formula of the form $\varphi(\bar{t})$ using rules (i) and (ii) of Definition 1.3 as well as applications of bound quantifiers. A $\boldsymbol{\Delta_0\langle P\rangle}$ **formula** is a $\Delta_0(P)$ formula that is positive in P.[1]

A $\boldsymbol{\Sigma_1(P)}$ **formula** (resp. $\boldsymbol{\Sigma_1\langle P\rangle}$ **formula**) is a formula of the form

$$\sup_{x_0} \sup_{x_1} \sup_{x_2} \ldots \varphi$$

for some sequence $\{x_i\}_{i<n}$ of distinct variables and with φ a $\Delta_0(P)$ formula (resp. $\Delta_0\langle P\rangle$ formula). A $\Pi_1(P)$ or $\Pi_1\langle P\rangle$ formula is the same but with infimum quantifiers instead of supremum quantifiers.

Note that when a and b are rational, a bound quantifier applied to a restricted formula yields a restricted formula. Note also that if φ is positive for P, then $\sup_{x<t} \varphi$ and $\inf_{x<t} \varphi$ are also positive for P.

Definition 1.9. For any $P, Q : \omega \to [0, 1]$ we define the following.

- We say that ***P* is analog enumeration reducible to *Q***, written $P \leq_{ae} Q$, if there is some $\Sigma_1\langle Q\rangle$ formula $\varphi(x)$ (with a single free variable x) such that for all $n < \omega$, $\varphi(n) = P(n)$.
- We say that ***P* is analog reducible to *Q***, written $P \leq_a Q$, if there is a $\Sigma_1(Q)$ formula $\varphi(x)$ and a $\Pi_1(Q)$ formula $\psi(x)$ such that for all $n < \omega$, $\varphi(n) = \psi(n) = P(n)$.

[1]The $\langle\rangle$ notation was chosen by analogy with Definition 3.1 in [1].

The equivalences $\equiv_{ae}$ and $\equiv_a$ and the induced degree structures $\mathscr{D}_{ae}$ and $\mathscr{D}_a$ are defined in the typical way. The ae-degree of Q is written $\mathbf{d}_{ae}(Q)$ and the a-degree of Q is written $\mathbf{d}_a(Q)$.

Definition 1.10. A **restricted connective expression** is any expression generated by 1, $x+y$, $x \wedge y$, $x \vee y$, and $x \mapsto r \cdot x$ for $r \in \mathbb{Q}$ on any variables.

A restricted connective expression is in **normal form** if it is a maximum of minimums of affine combinations of variables and constants (i.e. $r \cdot 1$).

We define the concept of a restricted connective expression being **positive in *V*** for some set of variables V inductively.

- Any variable is positive in V.
- If F and G are positive in V, then 1, $F+G$, $F \wedge G$, $F \vee G$, and $r \cdot F$ for any $r \geq 0$ are all positive in V.

Lemma 1.11. *If $F : \mathbb{R}^n \to \mathbb{R}$ is a partial function that is continuous on the set $X = \prod_{i<n} I_i$ for some sequence of intervals I_i and some $n \leq \omega$ and furthermore F is positive for the inputs A on X, where A is some set of indices $< n$, then there exists a sequence $\{F_i\}_{i<\omega}$ of restricted connective expressions in the variables $\{x_i\}_{i<n}$ which are positive in $V = \{x_i\}_{i \in A}$ such that $\{F_i\}$ uniformly converges to F on X monotonically from below.*

Proof. For any $\bar{a} \in \mathbb{R}^n$ and finite $k \leq n$, define

$$G_{\bar{a},k}(\bar{x}) = \min_{k \geq i \notin A} |x_i - a_i| \wedge \min_{k \geq i \in A} ((x_i - a_i) \vee 0),$$

where $|x_i - a_i| = (x_i - a_i) \vee (a_i - x_i)$. Note that this is a restricted connective expression that is positive in V when all of the a_i are rational. Also note that by construction $G_{\bar{a},k}(\bar{a}) = 0$ and $G_{\bar{a},k}(\bar{c}) \leq 0$ for all $\bar{c}$.

Claim: For any $\bar{a} \in X$ and any $b < F(\bar{a})$, there is a finite $k \leq n$ and an $m > 0$ such that $b + mG_{\bar{a},k}(\bar{x}) < F(\bar{x})$ for all $\bar{x} \in X$.

Proof of claim: For each $\bar{c} \in X$, if there is a $k_{\bar{c}} \notin A$ such that $c_{k_{\bar{c}}} \neq a_{k_{\bar{c}}}$ or there is a $k_{\bar{c}} \in A$ such that $c_{k_{\bar{c}}} < a_{k_{\bar{c}}}$, then there exists an $m_{\bar{c}} > 0$ large enough such that $b + m_{\bar{c}} G_{\bar{a},k_{\bar{c}}}(\bar{c}) < F(\bar{c})$. This must be true for all $\bar{c}'$ in some open neighborhood $U_{\bar{c}}$ of $\bar{c}$.

On the other hand if $c_k = a_k$ for all $k \notin A$ and $c_k \geq a_k$ for all $k \in A$, then since F is positive in A, we must have that $F(\bar{c}) \geq F(\bar{a}) > b$, so we must have $F(\bar{c}) > b + mG_{\bar{a},k}(\bar{c})$ for any $m > 0$ and finite $k \leq n$. Set $m_{\bar{c}} = 1$ and $k_{\bar{c}} = 1$. Again the inequality must be true in some open neighborhood $U_{\bar{c}}$ of $\bar{c}$.

Now by compactness there is a finite set $X_0 \subset X$ such that $\bigcup_{\bar{c} \in X_0} U_{\bar{c}} \supseteq X$. So now let $m_{\bar{a}} = \max_{\bar{c} \in X_0} m_{\bar{c}}$ and let $k_{\bar{a}} = \max_{\bar{c} \in X_0} m_{\bar{c}}$. Then by construction we have that $b + m_{\bar{a}} G_{\bar{a},k_{\bar{a}}}(\bar{x}) < F(\bar{x})$ for all $\bar{x} \in X$. $\square_{\text{claim}}$

In particular, by modifying the relevant values of a_i and m slightly so that they are rational, what we get is that for any $\bar{a} \in X$ and any $\varepsilon > 0$, there exists a restricted connective expression H that is positive in V such that $H(\bar{x}) < F(\bar{x})$ for all $\bar{x} \in X$ and such that $F(\bar{a}) < H(\bar{a}) + \varepsilon$. Fix $\varepsilon = 2^{-\ell}$ and let $H_{\bar{a}}$ be such a restricted connective expression for each $\bar{a} \in X$. There must exist an open neighborhood $V_{\bar{a}}$ of $\bar{a}$ such that for all $\bar{c} \in V_{\bar{a}}$, $F(\bar{c}) < H(\bar{c}) + \varepsilon$. By compactness there is a finite set $X_1 \subseteq X$ such that $\bigcup_{\bar{c} \in X_1} V_{\bar{c}} \supseteq X$. This implies that $F_\ell = \max_{\bar{c} \in X_1} H_{\bar{c}}$ is a restricted connective expression that is positive in V satisfying $F_\ell(\bar{x}) < F(\bar{x}) < F_\ell(\bar{x}) + 2^{-\ell}$ for all $\bar{x} \in X$.

By compactness there exists a sub-sequence of $\{F_\ell\}_{\ell<\omega}$ that is monotonically increasing. This is the required sequence. $\square$

Corollary 1.12. *If φ is a $\Delta_0\langle P\rangle$ formula then for any $\varepsilon > 0$ there is a restricted $\Delta_0\langle P\rangle$ formula ψ such that $|\varphi(n) - \psi(n)| < \varepsilon$ for all $n < \omega$ and all choices of $P : \omega \to [0,1]$. The same is true of $\Delta_0(P)$ formulas.*

Proof. We need to prove this by induction on the complexity of $\Delta_0\langle P\rangle$ formulas.

Case 1: φ is $Q\bar{t}$ for some atomic predicate Q and sequence of terms $\bar{t}$.

Let $\psi = Q\bar{t}$.

Case 2: φ is $F\bar{\chi}$ for some sequence of $\Delta_0\langle P\rangle$ formulas $\{\chi_i\}_{i<n}$, with $n \leq \omega$, and F positive in the indices V, where V is the set of $i < n$ such that χ_i contains the predicate symbol P.

Assume that the result has been shown for all of the χ_i. Find a $\delta > 0$ and a $k < \omega$ such that for any $\bar{a}, \bar{b} \in X = \prod_{i<n} I(\chi_i)$, if $|a_i - b_i| < \delta$ for all $i < k$, then $|F(\bar{a}) - F(\bar{b})| < \frac{1}{2}\varepsilon$. By Lemma 1.11 we can find a restricted connective expression G such that for all $\bar{a} \in X$, $|F(\bar{a}) - G(\bar{a})| < \frac{1}{2}\varepsilon$. By increasing k if necessary we may assume that G only contains variables of the form x_i for $i < k$. Now by the induction hypothesis for each $i < k$ we can find a restricted $\Delta_0\langle P\rangle$ formula η_i such that for any $n < \omega$ and any $P : \omega \to [0,1]$, $|\chi_i(n) - \eta_i(n)| < \delta$. Let $\psi = G(\bar{\eta})$. Clearly by construction ψ is a $\Delta_0\langle P\rangle$ formula. Furthermore for any $n < \omega$ and $P : \omega \to [0,1]$, we have that $|\varphi(n) - \psi(n)| \leq |F(\bar{\chi}) - F(\bar{\eta})| + |F(\bar{\eta}) - G(\bar{\eta})| < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon$, where $F(\bar{\eta})$ is understood to be F applied to the formulas η padded by arbitrary constant formulas that take on a value in the appropriate $I(\chi_i)$.

Case 3: φ is $\sup_{x<t} \chi$ or $\inf_{x<t} \chi$ and the result is already known for χ.

Let η be a restricted $\Delta_0\langle P\rangle$ formula that approximates χ to within ε in the sense of the result. Then let $\psi = \sup_{x<t} \eta$ or $\psi = \inf_{x<t} \eta$, respectively.

The proof for $\Delta_0(P)$ formulas is the same but easier. $\square$

Lemma 1.13. *For any fixed function $Q : \omega \to [0,1]$, the set of functions $P : \omega \to [0,1]$ that are $\Sigma_1\langle Q\rangle$ definable is closed in the uniform metric.*

Proof. Suppose that for every $n < \omega$ there is a $P_n : \omega \to [0,1]$ that is $\Sigma_1\langle Q\rangle$ definable with $|P - P_n| < 4^{-n}$. Let $\sup_{x_0} \sup_{x_1} \dots \varphi_n(\bar{x}, y)$ be the $\Sigma_1\langle Q\rangle$ formula that defines $P_n(y)$. By shifting these formulas if necessary we may assume that $P_n \leq P$ for every $n < \omega$. Furthermore by replacing these formulas with $(\varphi_n \vee 0) \wedge 1$ we may assume that their syntactic range is a subset of $[0,1]$.

Let $\bar{r}$ be an ω-tuple and define $\mathcal{F}^+_n(\bar{r})$ inductively:

- $\mathcal{F}^+_0(\bar{r}) = r_0$.
- $\mathcal{F}^+_{n+1}(\bar{r}) = \mathcal{F}^+_n(\bar{r})$ if $r_{n+1} \leq \mathcal{F}^+_n(\bar{r})$.
- $\mathcal{F}^+_{n+1}(\bar{r}) = r_{n+1}$ if $\mathcal{F}^+_n(\bar{r}) \leq r_{n+1} \leq \mathcal{F}^+_n(\bar{r}) + 2^{-n}$.
- $\mathcal{F}^+_{n+1}(\bar{r}) = \mathcal{F}^+_n(\bar{r}) + 2^{-n}$ if $\mathcal{F}^+_n(\bar{r}) + 2^{-n} \leq r_{n+1}$.

And then let $\mathcal{F}^+(\bar{r}) = \lim_{n\to\infty} \mathcal{F}^+_n(\bar{r})$. Note that this limit always exists. Furthermore note that this function is continuous on $[0,1]^\omega$ and is positive in every argument. ($\mathcal{F}^+$ is a slightly modified version of the 'forced limit' function from [5].)

Now consider the formula $\psi(\bar{x}^0, \bar{x}^1, \dots, y) = \mathcal{F}^+(\varphi_0(\bar{x}^0, y), \varphi_1(\bar{x}^1, y), \dots)$. Let $i(n), j(n)$ be some enumeration of ω^2 and consider the formula

$$\chi(y) = \sup_{x^{j(0)}_{i(0)}} \sup_{x^{j(1)}_{i(1)}} \dots \psi(\bar{x}^0, \bar{x}^1, \dots, y).$$

This is a $\Sigma_1\langle Q\rangle$ formula. All we need to do is argue that it evaluates to P. Fix $k<\omega$ and consider $\chi(k)$. For any $\varepsilon>0$ for each $n<\omega$ find $\{a_i^n\}_{i<\omega}$ such that $\varphi_n(\bar{a}^n,k)+\varepsilon>\sup_{\bar{x}}\varphi_n(\bar{x},k)$. Plugging these into the Δ_0 part of χ gives us that $\chi(k)\geq P(k)-\varepsilon$. Since we can do this for every $\varepsilon>0$ we have that $\chi(k)\geq P(k)$. But by shifting the φ_n down we ensured that $\chi(k)\leq P(k)$, so we have that $\chi(k)=P(k)$ for every $k<\omega$. □

Definition 1.14. Let $P,Q:\omega\to[0,1]$ be arbitrary functions. The **join of P and Q**, written $P\oplus Q$, is the function defined by $(P\oplus Q)(2n)=P(n)$ and $(P\oplus Q)(2n+1)=Q(n)$.

Proposition 1.15. *Let $P,Q:\omega\to[0,1]$ be arbitrary functions. P is $\Sigma_1(Q)$ definable if and only if P is $\Sigma_1\langle Q\oplus(1-Q)\rangle$ definable.*

Proof. Clearly if P is $\Sigma_1\langle Q\oplus(1-Q)\rangle$ definable, then it is $\Sigma_1(Q)$ definable. So assume that P is $\Sigma_1(Q)$ definable. Let $\varphi(\bar{x},y)$ be a $\Delta_1(Q)$ formula such that $\sup_{\bar{x}}\varphi(\bar{x},y)$ defines $P(y)$. Let $\varphi_n(\bar{x},y)$ be a sequence of restricted $\Delta_1(Q)$ formulas limiting to $\varphi(\bar{x},y)$. Assume that they are in normal form. Let $S=Q\oplus(1-Q)$. In each φ_n, replace with $r\cdot S(2t)$ each instance of $r\cdot Q(t)$ for which $r\geq 0$ holds and replace with $(-r)\cdot(S(2t+1)-1)$ each instance of $r\cdot Q(t)$ for which $r<0$ holds. The resulting formulas are positive in S and evaluate to the same values as the original formulas, so by Lemma 1.13, P is $\Sigma_1\langle S\rangle=\Sigma_1\langle Q\oplus(1-Q)\rangle$. □

Corollary 1.16. *Let $P,Q:\omega\to[0,1]$ be arbitrary functions. $P\leq_a Q$ if and only if $P\oplus(1-P)\leq_{ae} Q\oplus(1-Q)$.*

Proof. From the previous proposition it is obvious that if $P\leq_a Q$ then $P\oplus(1-P)\leq_{ae} Q\oplus(1-Q)$. Conversely if $P\oplus(1-P)\leq_{ae} Q\oplus(1-Q)$ then $P\leq_{ae} Q\oplus(1-Q)$, which implies that $P\leq_a Q\oplus(1-Q)\equiv_a Q$. □

So we have that the map $P\mapsto P\oplus(1-P)$ induces an embedding of the a degrees into the ae degrees, which we'll write $\iota_{a,ae}$. Some direct manipulation shows that this embedding preserves joins. We call an ae-degree 'total' if it is in the image of this map, or in other words if it is the degree of some $P\oplus(1-P)$.

Finally we can extend Lemma 1.13 to the other three relevant notions as well as demonstrate that these notions have a 'separable predecessor property.'

Corollary 1.17. *For any $Q:\omega\to[0,1]$ the following sets are all separable and closed in the uniform metric:*

- *The set of P that are $\Sigma_1\langle Q\rangle$.*
- *The set of P that are $\Sigma_1(Q)$.*
- *The set of P that are $\leq_{ae} Q$.*
- *The set of P that are $\leq_a Q$.*

Proof. Separability follows from the fact that (positive) restricted Σ_1 formulas are dense among (positive) Σ_1 formulas. Closedness follows from Lemma 1.13 and the previous corollary. □

Theorem 1.18. *Let $P,Q:\omega\to[0,1]$ be arbitrary functions.*

(i) $P\leq_{ae} Q$ if and only if for every $\varepsilon>0$ there exists a $\delta>0$ and an enumeration operator W such that for any $X\subseteq\omega$ such that $\|Q-Q'\|<\delta$, where $Q'(n)=\sup\{r\in\mathbb{Q}:\langle r,n\rangle\in X\}$, then if we set $P'(n)=\sup\{r\in\mathbb{Q}:$

$\langle r,n\rangle \in W^X\}$ we have that $\|P-P'\| \le \varepsilon$ (i.e. $P'(n)$ is a sequence that is uniformly lower semi-computable in X as a positive oracle).

(ii) $P \le_{ae} Q$ if and only if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for any $Q' : \omega \to [0,1]$ with $\|Q-Q'\| < \delta$ and any enumeration X of the set $\{\langle r,n\rangle : r \in \mathbb{Q} \wedge r \le Q'(n)\}$, there is a $P' : \omega \to [0,1]$ with $\|P-P'\| \le \varepsilon$ such that for some enumeration Y of $\{\langle r,n\rangle : r \in \mathbb{Q} \wedge r \le P'(n)\}$, $Y \le_T X$.

(iii) $P \le_a Q$ if and only if for every $\varepsilon > 0$ there exists a $\delta > 0$ and a Turing operator Φ (which we're taking to output rational numbers) such that for any $Q' : \omega \to \mathbb{Q}$ with $\|Q-Q'\| < \delta$, $\left\|P-\Phi^{Q'}\right\| \le \varepsilon$.

(iv) $P \le_a Q$ if and only if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for any $Q' : \omega \to \mathbb{Q}$ with $\|Q-Q'\| < \delta$ there is a $P' : \omega \to \mathbb{Q}$ with $\|P-P'\| \le \varepsilon$ such that $P \le_T Q$.

Proof. (i), $\Rightarrow$: Assume that $P \le_{ae} Q$. Let $\sup_{x_0}\sup_{x_1}\ldots\varphi(\bar{x},y,Q)$ be a $\Sigma_1\langle Q\rangle$ formula defining P. Fix $\varepsilon > 0$ and find a $\delta > 0$ such that for any Q' if $\|Q-Q'\| < \delta$, then $|\varphi(\bar{x},y,Q)-\varphi(\bar{x},y,Q')| < \frac{\varepsilon}{2}$. Find a restricted $\Delta_0\langle Q\rangle$ formula $\psi(\bar{x},y,Q)$ such that for any $\bar{x}$, y, and Q, $|\varphi(\bar{x},y,Q)-\psi(\bar{x},y,Q)| < \frac{\varepsilon}{2}$.

Now by construction we have that for all $\bar{x}$, y, and any $X \subseteq \omega$ with $\|Q-Q'\| < \frac{\delta}{2}$, where $Q'(n) = \sup\{r \in \mathbb{Q} : \langle r,n\rangle \in X\}$, we have $|\varphi(\bar{x},y,Q)-\psi(\bar{x},y,Q')| \le \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon$. This implies that for any y, $|\sup_{\bar{x}}\varphi(\bar{x},y,Q)-\sup_{\bar{x}}\psi(\bar{x},y,Q')| \le \varepsilon$. Now since ψ is restricted and positive, $P'(y) = \sup_{\bar{x}}\psi(\bar{x},y,Q')$ is uniformly lower semi-computable in X (as a positive oracle) as witnessed by a fixed enumeration operator that depends only on ψ.

(i), $\Leftarrow$: Assume that the right-hand side of the equivalence in *(i)* holds for some P and Q. Fix $\varepsilon > 0$ and find the corresponding $\delta > 0$ and enumeration operator W. Find an $N < \omega$ such that $2^{-N} < \frac{\delta}{2}$ and $2^{-N} \le \varepsilon$.

Let $\eta(x,y,z,w)$ be a discrete Δ_0 formula such that

$$\langle r_0,k_0\rangle, \langle r_1,k_1\rangle, \ldots, \langle r_m,k_m\rangle \in X \Rightarrow \langle s,\ell\rangle \in W^X$$

is a rule of the enumeration operator W if and only if

$$(\heartsuit) \qquad \exists x\eta(x, \langle\langle r_0,k_0\rangle, \langle r_1,k_1\rangle, \ldots, \langle r_m,k_m\rangle\rangle, s, \ell).$$

Furthermore we may assume that the enumeration operator W is 'monotonic' in the sense that if $(\heartsuit)$ holds then for any $r_i' \ge r_i$ and $s' \le s$, $\exists x\eta(x, \langle\langle r_0',k_0\rangle, \langle r_1',k_1\rangle, \ldots, \langle r_m',k_m\rangle\rangle, s', \ell)$ holds as well.

Let $R(z,\sigma,j,r)$ be a discrete Δ_0 formula that is true if and only if z is sufficiently large and $\sigma = \langle\langle r_0,k_0\rangle, \langle r_1,k_1\rangle, \ldots, \langle r_m,k_m\rangle\rangle$ with $r_j = r$. Let $K(z,\sigma,j,k)$ be a discrete Δ_0 formula that is true if and only if z is sufficiently large and $\sigma = \langle\langle r_0,k_0\rangle, \langle r_1,k_1\rangle, \ldots, \langle r_m,k_m\rangle\rangle$ with $k_j = k$. Let $L(z,\sigma,m)$ be a discrete Δ_0 formula that is true if and only if z is sufficiently large and $\sigma = \langle\langle r_0,k_0\rangle, \langle r_1,k_1\rangle, \ldots, \langle r_m,k_m\rangle\rangle$ (i.e. $m+1$ is the length of σ).

For any $0 < i \le 2^N$, let $f_i(x) = \left(2^N(x-(i-1)2^{-N}) \vee 0\right) \wedge 1$. Note that f_i always takes on values in $[0,1]$, is positive in x, and has $f_i((i-1)2^{-N}) = 0$ and $f_i(i2^{-N}) = 1$. Let $f_0(x) = 1$.

Now let

$$\psi_i = \inf_{j \le m}\sup_{k<z}\bigvee_{\ell \le 2^N}[R(z,\sigma,j,\ulcorner \ell 2^{-N}\urcorner) \wedge K(z,\sigma,j,k)] \wedge f_\ell(Q(k)),$$

$$\varphi(x,z,\sigma,m,y) = [L(z,\sigma,m)] \wedge \bigvee_{i\leq 2^N} i2^{-N}\left(\left[\eta(x,\sigma,i2^{-N},y)\right] \wedge \psi_i\right).$$

Note that φ is $\Delta_0\langle Q\rangle$. Our goal is to consider $P''(y) = \sup_{x,z,\sigma,m}\varphi(x,z,\sigma,m,y)$, which is a $\Sigma_1\langle Q\rangle$ formula, and to show that it approximates $P(y)$. To unpack what φ means, recall that since we're taking 0 to be false and 1 to be true, inf functions like $\forall$ and sup functions like $\exists$, so ψ_i evaluates to zero unless for every $j \leq m$ there exists $k < z$ and $\ell \leq 2^N$ such that the jth term in σ is $\langle \ulcorner \ell 2^{-N}\urcorner, k\rangle$ and such that $(\ell-1)2^{-N} < Q(k)$, in which case it evaluates to $\left(1 \wedge 2^N \bigwedge_{j\leq m}(Q(k_j)-r_j)\right) \vee 0$. In particular if $Q(k_j) \geq r_j$ for each $j \leq m$, then φ evaluates to 1.

So now we can say that $\varphi(x,z,\sigma,m,y)$ evaluates to 0 unless z is sufficiently large and x corresponds (via η) to a rule in the enumeration operator of the form

$$\langle a_0 2^{-N}, k_0\rangle, \langle a_1 2^{-N}, k_1\rangle, \ldots, \langle a_m 2^{-N}, k_m\rangle \in X \Rightarrow \langle i2^{-N}, y\rangle \in W^X.$$

In this case, it takes a value in the interval $[0, i2^{-N}]$. More specifically, if for each $j \leq m$, $Q(k_j) \leq (a_j - 1)2^{-N}$, then $\varphi(x,z,\sigma,m,y)$ evaluates to 0 and if for each $j \leq m$, $a_j 2^{-N} \leq Q(k_j)$, then $\varphi(x,z,\sigma,m,y)$ evaluates to $i2^{-N}$.

So let $Q' : \omega \to [0,1]$ be a function satisfying $\|Q - Q'\| < \frac{\delta}{2}$ and let $X = \{\langle r,n\rangle : r \in \mathbb{Q} \wedge r \leq Q'(n)\}$. Let $Q'_\downarrow$ be Q' with each output rounded down to the nearest multiple of 2^{-N} and let $Q'_\uparrow$ be Q' with each output rounded up to the nearest multiple of 2^{-N}. $Q'_\downarrow$ and $Q'_\uparrow$ both have uniform distance $< \frac{\delta}{2} + \frac{\delta}{2} = \delta$ from Q, so if $X_\downarrow$, $X_\uparrow$ are all defined in corresponding ways and we let $P'_v(n) = \sup\{r \in \mathbb{Q} : \langle r,n\rangle \in W^{X_v}\}$ for $v \in \{\downarrow,\uparrow\}$, then we have that $P'_\downarrow$ and $P'_\uparrow$ have uniform distance $\leq \varepsilon$ from P.

Let $P''(y) = \sup_{x,z,\sigma,m}\varphi(x,z,\sigma,m,y,Q')$, and let $P''_\downarrow$ and $P''_\uparrow$ be defined similarly.

Claim: For any $Q^\dagger : \omega \to \{0, 2^{-N}, 2\cdot 2^{-N}, \ldots, 1\}$, if $Y = \{\langle r,n\rangle : r \in \mathbb{Q} \wedge r \leq Q'(n)\}$, $P^\dagger(y) = \sup\{r \in \mathbb{Q} : \langle r,n\rangle \in Y\}$, and $P^{\dagger\dagger}(y) = \sup_{x,z,\sigma,m}\varphi(x,z,\sigma,m,y,Q^\dagger)$, then $P^\dagger - 2^{-N} \leq P^{\dagger\dagger} \leq P^\dagger$.

Proof of claim: Given the restriction that $Q^\dagger$ takes on values in $\{0, 2^{-N}, 2\cdot 2^{-N}, \ldots 1\}$, our description of φ gives the behavior of $P^{\dagger\dagger}$ completely (the intermediate cases never occur), and what we have is that $P^{\dagger\dagger}(n)$ is always either $P^\dagger(n)$ rounded down to the nearest multiple of 2^{-N} or either $P^\dagger(n)$ or $P^\dagger(n) - 2^{-N}$ if $P^\dagger(n)$ is itself a multiple 2^{-N} (this relies on the monotonicity property we required W to have), which establishes the required inequality. $\square_{\text{claim}}$

As a consequence of this claim we have that $\|P''_v - P'_v\| \leq 2^{-N} \leq \varepsilon$ for $v \in \{\downarrow,\uparrow\}$, hence by positivity we have that $P''_\downarrow \leq P'' \leq P''_\uparrow$, so we have that $P'_\downarrow - \varepsilon \leq P'' \leq P'_\uparrow + \varepsilon$. This implies that $\|P - P''\| \leq 2\varepsilon$.

Thus there are $P'' \leq_{ae} Q$ arbitrarily close to P in the uniform metric, so by Lemma 1.13 we have that $P \leq_{ae} Q$ as well, as required.

(ii): The $\Rightarrow$ direction is clear from part *(i)*, so assume that the right-hand side of *(ii)* holds for some P and Q. Fix $\varepsilon > 0$ and find the corresponding $\delta > 0$ given by the right-hand side of *(ii)*.

We want to try to build a generic enumeration X of the set $\{\langle r,n\rangle : r \in \mathbb{Q} \wedge r \leq Q'(n)\}$ for some $Q' : \omega \to [0,1]$ satisfying $\|Q - Q'\| \leq \frac{\delta}{2}$. Proceed in stages, where X_s is the initial segment of the enumeration enumerated by stage s.

Given a partial enumeration X_s, say that an extension $X_{s+1} \succ X_s$ is 'good for γ' if for any new term of the form $\langle r,n\rangle$ enumerated, $r \leq Q(n) + \gamma$.

On stage $s+1 = 2e$, try to diagonalize against the Turing operator with index e. Specifically, look to see if there's an extension $X_{s+1} \succ X_s$ that is good for $2^{-s-1}\delta$ such that either

(1) there exists an m such that $\Phi_e^{X_{s+1}}(m)$ halts and enumerates $\langle r, n\rangle$ for some $r \geq P(n) + 2\varepsilon$, or
(2) there exists an n such that for any extension $Y \succ X_{s+1}$ that is good for $2^{-s-2}\delta$ and any m, $\Phi_e^Y(m)$ does not enumerate $\langle r, n\rangle$ for some $r > P(n)-2\varepsilon$.

If either of these are possible, let that be X_{s+1}. If neither of these are possible, stop, as the construction has failed.

On stage $s + 1 = 2n + 1$, enumerate some pair of the form $\langle r, n\rangle$ with $r \in [Q(n) - 2^{-s-1}\delta, Q(n) + 2^{-s-1}\delta] \cap \mathbb{Q}$. (Note that this extension is necessarily good for $2^{-s-1}\delta$.)

Assume that the construction succeeds in building some enumeration X. Let $Q'(n) = \sup\{r \in \mathbb{Q} : \langle r, n\rangle \text{ is enumerated by } X\}$. By construction we have that $\|Q - Q'\| \leq \frac{\delta}{2} < \delta$, so by assumption we have that there is some index e such that Φ_e^X computes an enumeration of $\{\langle r, n\rangle : r \in \mathbb{Q} \wedge r \leq P'(n)\}$ for some $P' : \omega \to [0, 1]$ with $\|P - P'\| \leq \varepsilon < 2\varepsilon$.

But now we have a contradiction at stage $s + 1 = 2e$. Since the construction succeeded, either there is an m such that $\Phi_e^X(m)$ halts and enumerates $\langle r, n\rangle$ for some $r \geq P(n) + 2\varepsilon$, or for every m, $\Phi_e^X(m)$ does not enumerate $\langle r, n\rangle$ for any $r > P(n) - 2\varepsilon$, both of which imply that $\|P - P'\| \geq 2\varepsilon > \varepsilon$.

Therefore the construction must fail at some stage $s + 1 = 2e$ with some partial enumeration X_s built. This means that for every extension $Z \succeq X_s$ that is good for $2^{-s-1}\delta$,

($*$) for every m, if $\Phi_e^Z(m)$ halts and enumerates $\langle r, n\rangle$ then $r < P(n) + 2\varepsilon$, and
($**$) for every n there exists an extension $Y \succ Z$ that is good for $2^{-s-2}\delta$ and an m such that $\Phi_e^Y(m)$ enumerates $\langle r, n\rangle$ for some $r > P(n) - 2\varepsilon$.

Find a rational number $t > 0$ such that $(2^{-s-2} + 2^{-s-3})\delta < t < (2^{-s-2} + 2^{-s-3} + 2^{-s-4})\delta$.

Let W be an enumeration operator described by the following rules (we allow duplicates in the list)

$$\langle r_0, n_0\rangle, \langle r_1, n_1\rangle, \ldots, \langle r_k, n_k\rangle \in Y \Rightarrow \langle s, m\rangle \in W^Y$$

whenever there exists

- a sequence $\{r'_i\}_{i\leq k}$ of rational numbers satisfying $r'_i \leq r_i + t$ for each $i \leq k$,
- a listing Y of $\langle r'_0, n_0\rangle, \langle r'_1, n_1\rangle, \ldots, \langle r'_k, n_k\rangle$, and
- an m, such that $\Phi_e^{X_s \frown Y}(m)$ halts and enumerates $\langle s, m\rangle$.

This is clearly a c.e. list of rules, so W is an enumeration operator.

Let $Q' : \omega \to [0, 1]$ satisfy $\|Q - Q'\| < 2^{-s-4}\delta$. Let $X' = \{\langle r, n\rangle : r \in \mathbb{Q} \wedge r \leq Q'(n)\}$. Let $P'(n) = \sup\{r \in \mathbb{Q} : \langle r, n\rangle \in W^{X'}\}$. We would like to argue that $\|P - P'\| \leq 2\varepsilon$.

For each n, if $W^{X'}$ enumerates $\langle r, n\rangle$, then by the choice of t, any extension $X_s \frown Y$ (as in the definition of W) is good for $(2^{-s-2} + 2^{-s-3} + 2^{-s-4} + 2^{-s-4})\delta = 2^{-s-1}\delta$, so by ($*$), we have that $r < P(n) + 2\varepsilon$. This implies that $P'(n) \leq P(n) + 2\varepsilon$.

On the other hand, by the choice of t, the search in the definition of W covers all extensions $X_s \frown Y$ that are good for $2^{-s-2}\delta$ since $2^{-s-2}\delta < (2^{-s-2} + 2^{-s-3} - 2^{-s-4})\delta < t - 2^{-s-4}\delta$, so $W^{X'}$ will enumerate some $\langle r, n\rangle$ with $r > P(n) - 2\varepsilon$.

Together this implies that $|P(n) - P'(n)| \leq 2\varepsilon$ for every n, so in particular $\|P - P'\| \leq 2\varepsilon$.

Since we can do this for any $\varepsilon > 0$, we have that the right-hand side of *(i)* holds with P and Q, so by part *(i)* we have that $P \leq_{ae} Q$, as required.

(iii), $\Rightarrow$: By the same argument as in part *(i)* for the $\Rightarrow$ direction, we get that for any $\varepsilon > 0$ there is a $\delta > 0$ and two enumeration operators W_+ and W_- (coming from restricted formulas approximating the $\Sigma_1(Q)$ and $\Pi_1(Q)$ definitions of P) such that for any Q' with $\|Q - Q'\| < \delta$, $W_+^{Q' \oplus (1-Q')}$ enumerates lower bounds to a P'_+ with $\|P - P'_+\| \leq \frac{\varepsilon}{5}$ and $W_-^{Q' \oplus (1-Q')}$ enumerates upper bounds to a P'_- with $\|P - P'_-\| \leq \frac{\varepsilon}{5}$ (we may assume that ε is rational by shrinking it if necessary). Our actual Turing operator Φ will use W_+ and W_-. For a given n once $W_+^{Q'}$ enumerate a lower bound r on P'_+ and $W_-^{Q'}$ enumerates an upper bound s on P'_- such that $|r - s| < \frac{\varepsilon}{2}$ then we output r as the value of $\Phi^{Q'}(n)$. This process must always halt and will always return an output that is within $\frac{\varepsilon}{2} + \frac{2\varepsilon}{5} < \varepsilon$ of the value of $P(n)$, as required.

(iv): The $\Rightarrow$ direction is clear from the $\Rightarrow$ direction of part *(iii)*, so assume that the right-hand side of *(iv)* holds for some P and Q. We will show that $P \oplus (1-P) \leq_{ae} Q \oplus (1-Q)$ and use Corollary 1.16. Assume the right-hand side *(iii)* holds for some P and Q. Fix $\varepsilon > 0$ and find $\delta > 0$ according to the right-hand side of *(ii)*. Assume that δ is rational, shrinking it if necessary. Now assume that we've been given $S : \omega \to [0,1]$ such that $\|S - Q \oplus (1 - Q)\| < \frac{\delta}{3}$. This implies that for every n, $|S(2n) - (1 - S(2n+1))| < \frac{2\delta}{3}$. Also assume we're given some enumeration of $\{\langle r, n\rangle : r \in \mathbb{Q} \wedge r \leq S(n)\}$. For each n, wait until the enumeration enumerates a lower bound s_0 for $S(2n)$ and an upper bound s_1 for $S(2n+1)$ such that $|s_1 - s_0| \leq \frac{2\delta}{3}$ (this must happen eventually), then output $Q'(n) = s_0$. We have ensured that $\|P - P'\| < \frac{\delta}{3} + \frac{2\delta}{3} = \delta$. Now by the right-hand side of *(iv)*, there is some P' computable from Q' such that $\|P - P'\| \leq \varepsilon$. Then output an enumeration of $\{\langle r, n\rangle : r \in \mathbb{Q} \wedge r \leq (P \oplus (1 - P))(n)\}$.

Since we can do this for any $\varepsilon > 0$, part *(ii)* implies that $P \oplus (1 - P) \leq_{ae} Q \oplus (1 - Q)$. Thus by Corollary 1.16 $P \leq_a Q$, as required.

(iii), $\Leftarrow$: This is a direct corollary of part *(iv)*. □

These definitions are just for a restatement of parts *(ii)* and *(iv)* of Theorem 1.18.

Definition 1.19. For any $P : \omega \to [0,1]$, let $\mathcal{AE}(P)$ be the set of Turing degrees $\mathbf{a}$ such that P is in the uniform metric closure of

$$\{Q \in [0,1]^\omega : Q \text{ is uniformly lower semi-computable in } \mathbf{a}\},$$

and let $\mathcal{A}(P)$ be the set of Turing degrees $\mathbf{A}$ such that P is in the uniform metric closure of

$$\{Q \in [0,1]^\omega : Q \leq_T \mathbf{a}\}.$$

Corollary 1.20. *Let $P, Q : \omega \to [0,1]$ be arbitrary functions.*

(i) $P \leq_{ae} Q$ if and only if $\mathcal{AE}(P) \supseteq \mathcal{AE}(Q)$.
(ii) $P \leq_a Q$ if and only if $\mathcal{A}(P) \supseteq \mathcal{A}(Q)$.

In particular, $\mathcal{AE}$ and $\mathcal{A}$ embed $\mathscr{D}_{ae}$ and $\mathscr{D}_a$ into the Muchnik degrees.

Proof. *(i)* follows immediately from Theorem 1.18 part *(ii)*. *(ii)* largely follows from part *(iv)* of that theorem, but the only point to note is that given a function $Q : \omega \to [0, 1]$ computable in $\mathbf{a}$, there are always arbitrarily good rational approximations of Q also computable in $\mathbf{a}$. $\square$

Note the similarity of Corollary 1.20 with the following characterization of the continuous degrees: For any $P, Q \in [0, 1]^\omega$, $P \leq_r Q$ if and only if $\{\mathbf{a} : Q \leq_T \mathbf{a}\} \subseteq \{\mathbf{a} : P \leq_T \mathbf{a}\}$. The analogous condition with semi-computability gives an odd presentation of the enumeration degrees.

Now we come to a major difference between the analog degrees and their discrete counterparts. The *ae*-degrees and *a*-degrees both have ω-joins.

Proposition 1.21. *If $\{P_i\}_{i<\omega}$ is a sequence of functions $\omega \to [0, 1]$, then the ae-degree (resp. a-degree) of $\bigoplus_{i<\omega} 2^{-i}P_i$ is the least upper bound of the ae-degrees (resp. a-degrees) of the P_i.*

Proof. Clearly the *ae*/*a*-degree of $Q = \bigoplus_{i<\omega} 2^{-i}P_i$ is above the *ae*/*a*-degrees of each P_i, so we just need to show that any degree that is above all the P_i is also above Q. If we set $Q_n = \bigoplus_{i<\omega} 2^{-i}[i < n]P_i$, where $[i < n] = 1$ when $i < n$ and 0 otherwise, then the degrees of the Q_n are clearly below any degree above all of the P_i. Since these uniformly limit to Q, the degree of Q is below any degree that is above all of the P_i. $\square$

Definition 1.22. If $\{\mathbf{a}_i\}_{i<\omega}$ is a sequence of *ae*-degrees of the oracles $\{P_i\}$, then $\bigvee_{i<\omega} \mathbf{a}_i$ is the degree of $\bigoplus_{i<\omega} 2^{-i}P_i$. Likewise for a sequence of *a*-degrees.

Clearly the ω-join is preserved in the natural embedding of $\mathscr{D}_a$ into $\mathscr{D}_{ae}$.

Corollary 1.23. *If A is a set of functions $\omega \to [0, 1]$ that is separable in the uniform metric, then the set of ae-degrees (resp. a-degrees) of elements of A has a least upper bound.*

Proof. Let A_0 be a countable dense subset of A. The ω-join of the degrees of A_0, $\mathbf{a}$, has degree above every element of A_0. Since A_0 is dense in A, every element of A has degree below $\mathbf{a}$. $\square$

Corollary 1.24. *Any non-empty set of ae-degrees (resp. a-degrees) has a greatest lower bound. Any bounded set of ae-degrees (resp. a-degrees) has a least upper bound.*

Proof. This follows from the fact that the *ae*-degrees and *a*-degrees have the 'separable predecessor property,' as in the set of functions with degree $\leq$ any given degree is always separable in the uniform metric. $\square$

Finally there are natural definitions of jumps on the *ae*-degrees and the *a*-degrees.

Definition 1.25. Let $\{\varphi_i(x, Q)\}_{i<\omega}$ be an enumeration of all restricted $\Sigma_1\langle Q\rangle$ formulas. Let $\psi(x, Q)$ be a $\Sigma_1\langle Q\rangle$ formula such that $\psi(\langle i, n\rangle, Q) = 2^{-i}\varphi_i(n, Q)$.[2] For any $P : \omega \to [0, 1]$, let $\Psi(P) : \omega \to [0, 1]$ be such that $\Psi(P)(n) = \psi(n, P)$.

[2]Note that the 2^{-i} is necessary, otherwise ψ wouldn't be uniformly continuous in Q.

(i) The ***ae*-jump of a**, written $\mathbf{a}'$, where $\mathbf{a} = \mathbf{d}_{ae}(Q)$, is

$$\mathbf{d}_{ae}(\Psi(Q) \oplus (1 - \Psi(Q))).$$

(ii) The ***a*-jump of a**, written $\mathbf{a}'$, where $\mathbf{a} = \mathbf{d}_{a}(Q)$, is

$$\mathbf{d}_{a}(\Psi(Q \oplus (1 - Q))).$$

Iterated jumps are written in the typical way (e.g. $\mathbf{a}^{(17)}$).

Provided that these jumps are well defined on degrees, it is clear that the jump commutes with the natural inclusion of $\mathscr{D}_a$ into $\mathscr{D}_{ae}$ and that the jump of any ae-degree is total. To see that they are well defined, note that the jump of an ae-degree $\mathbf{a}$ can be characterized as the smallest total degree $\mathbf{b}$ such that for any $P \leq_{ae} \mathbf{a}$, $P \leq_a \mathbf{b}$. Likewise, almost by definition, the jump of an a-degree $\mathbf{a}$ is the least a-degree $\mathbf{b}$ such that for any $P \leq_{ae} \mathbf{a}$ (i.e. any P that is Σ_1 in $\mathbf{a}$), $P \leq_a \mathbf{b}$. Unlike in the discrete degrees, both jumps have fixed points, namely degrees of the form $\bigvee_{i<\omega} \mathbf{a}^{(i)}$. But from the point of view that the ae-degrees and a-degrees are more correctly thought of generalizations of the semi-lattices of countable ideals of e-degrees and Turing degrees, respectively, this is not very surprising, as it is analogous to the fact that countable jump ideals exist.

2. Discrete Degrees

Given the original definition of ae- and a-reducibility in terms of Σ_1 definability and the fact that continuous logic is in some sense a 'conservative' extension of discrete logic for discrete structures, it seems likely that when restricted to $\{0,1\}$-valued oracles, ae- and a-reducibility should coincide with enumeration and Turing reducibility. We will show that this is true in this section.

Definition 2.1. Let $\mathscr{D}_{ei}$ be the collection of countable enumeration ideals. For $a, b \in \mathscr{D}_{ei}$, let $a \leq_{ei} b$ mean $a \subseteq b$. Likewise, let $\mathscr{D}_{Ti}$ be the collection of countable Turing ideals, with $\leq_{Ti}$ defined analogously. In both cases we consider them to have a 'jump' wherein the jump of an ideal is the ideal generated by the pointwise jump of the ideal.

For any sequence of functions $\{X_i\}_{i<\omega}$, with $X_i : \omega \to [0,1]$, let $\mathbf{d}_{ei}(X_i)$ and $\mathbf{d}_{Ti}(X_i)$ be the enumeration and Turing ideals, respectively, generated by the X_i.

Let $\iota_{T,e} : \mathscr{D}_T \to \mathscr{D}_e$ be the inclusion map induced by the set map $X \mapsto X \oplus \overline{X}$.

Let $\iota_{Ti,ei} : \mathscr{D}_{Ti} \to \mathscr{D}_{ei}$ be the map induced by applying $\iota_{T,e}$ elementwise and closing under $\leq_e$.

For $x \in \{e, T\}$, let $\iota_{x,xi} : \mathscr{D}_x \to \mathscr{D}_{xI}$ be the map induced by passing to the downwards cone under $\leq_x$.

Every arrow $A \to B$ in a diagram in this paper will be an inclusion map of join semi-lattices, preserving jumps and least elements, written $\iota_{A,B}$, so we will omit arrows to make the diagrams easier to read. It is hardly worth remarking that the following diagram commutes.

$$\begin{array}{ccc} \mathscr{D}_T & \longrightarrow & \mathscr{D}_{Ti} \\ \downarrow & & \downarrow \\ \mathscr{D}_e & \longrightarrow & \mathscr{D}_{ei} \end{array}$$

And that the inclusion of $\mathscr{D}_{Ti}$ into $\mathscr{D}_{ei}$ preserves ω-joins. This diagram extends in a very natural way to include $\mathscr{D}_{ae}$ and $\mathscr{D}_a$.

Proposition 2.2. *Let $\{X_i\}_{i<\omega}$ and $\{Y_i\}_{i<\omega}$ be sequences of functions $X_i, Y_i : \omega \to \{0,1\}$.*

(i) $\mathbf{d}_{ei}(X_i) \leq_{ei} \mathbf{d}_{ei}(Y_i)$ *if and only if* $\bigoplus_{i<\omega} 2^{-i} X_i \leq_{ae} \bigoplus_{i<\omega} 2^{-i} Y_i$.
(ii) $\mathbf{d}_{Ti}(X_i) \leq_{Ti} \mathbf{d}_{Ti}(Y_i)$ *if and only if* $\bigoplus_{i<\omega} 2^{-i} X_i \leq_{a} \bigoplus_{i<\omega} 2^{-i} Y_i$.

Consequently, the map $\{X_i\} \mapsto \bigoplus_{i<\omega} 2^{-i} X_i$ *induces inclusion maps,* $\iota_{ei,ae} : \mathscr{D}_{ei} \to \mathscr{D}_{ae}$ *and* $\iota_{Ti,a} : \mathscr{D}_{Ti} \to \mathscr{D}_a$*, that preserve ω-joins, jumps, and* $\mathbf{0}$*, such that the following diagram commutes.*

$$\begin{array}{ccccc} \mathscr{D}_T & \longrightarrow & \mathscr{D}_{Ti} & \longrightarrow & \mathscr{D}_a \\ \downarrow & & \downarrow & & \downarrow \\ \mathscr{D}_e & \longrightarrow & \mathscr{D}_{ei} & \longrightarrow & \mathscr{D}_{ae} \end{array}$$

Furthermore for any $Q : \omega \to \{0,1\}$, $\iota_{ei,ae} \circ \iota_{e,ei} \circ \mathbf{d}_e(Q) = \mathbf{d}_{ae}(Q)$ *and* $\iota_{Ti,a} \circ \iota_{T,Ti} \circ \mathbf{d}_T(Q) = \mathbf{d}_a(Q)$.

Proof. (i),(ii): These both follow fairly immediately from Theorem 1.18 and the fact that for $x \in \{e, T\}$, $\mathbf{d}_{xi}(X_i) \leq_{xi} \mathbf{d}_{xi}(Y_i)$ if and only if for every n there is an m such that $\bigvee_{i<n} \mathbf{d}_x(X_i) \leq_x \bigvee_{i<m} \mathbf{d}_x(Y_i)$.

The induced inclusion maps clearly preserve $\mathbf{0}$. Preservation of ω-joins and the fact that the diagram commutes can be verified by direct manipulations of the oracle level definitions of the ω-joins and the embeddings.

To verify that the jump is preserved, note that for $x \in \{e, T\}$, $\mathbf{d}_{xi}(X_i)' = \mathbf{d}_{xi}\left((\oplus_{j<i} X_j)'\right)$ and that for any i, there are explicit fixed columns of the jump for $\mathscr{D}_{ae}$ or $\mathscr{D}_a$ that are equal to (scaled versions of) $(\oplus_{j<i} X_j)'$ (in whichever sense of the jump is relevant).

The fact that the e- or Turing degree of a $\{0,1\}$-valued oracle embedded into the ae- or a-degrees (respectively) is the same as its ae- or a-degree directly follows from the fact that for $(x, y) \in \{(e, ae), (T, a)\}$, the embedding $\iota_{xi,y} \circ \iota_{x,xi}$ also follows immediately from Theorem 1.18 (a $\{0,1\}$-valued oracle has the same ae- or a-degree as any non-uniform ω-join of the set of $\{0,1\}$-valued oracles that are ae- or a-reducible to it, respectively). ☐

In light of the previous proposition, we'll call an ae-degree or a-degree 'discrete' if it is the degree of a $\{0,1\}$-valued oracle, or in other words if it is in the image of the inclusion of $\mathscr{D}_e$ into $\mathscr{D}_{ae}$ or $\mathscr{D}_T$ into $\mathscr{D}_a$, respectively.

3. Continuous Degrees

The continuous degrees, $\mathscr{D}_r$, introduced by Miller in [8], correspond to a natural notion of reducibility, called 'representation reducibility,' between elements of computable metric spaces, although all continuous degrees occur as degrees of elements of the Hilbert cube, $[0,1]^\omega$, with the product metric. Here we'll give an equivalent definition for elements of the Hilbert cube.

Definition 3.1. Fix $Q \in [0,1]^\omega$. A **name of Q** is a function $f : \omega \to \mathbb{Q}^{<\omega}$ such that for each n and each $i < n$, $|f(n)(i) - Q(n)| < 2^{-n}$.

$P \in [0,1]^\omega$ is **representation reducible to Q**, written $P \leq_r Q$, if every name of Q computes a name of R. The induced degree structure is written $\mathscr{D}_r$. We'll let $\mathscr{D}_{ri}$ be the collection of countable ideals of elements of $\mathscr{D}_r$, ordered under inclusion.

$\iota_{T,r}$ is the inclusion map induced by the natural inclusion of 2^ω in $[0,1]^\omega$. $\iota_{r,e}$ is the inclusion map induced by mapping P to $\{\langle r,n,0\rangle : r \in \mathbb{Q} \wedge r < P(n)\} \cup \{\langle r,n,1\rangle : r \in \mathbb{Q} \wedge r > P(n)\}$, treated as a positive oracle. $\iota_{Ti,ri}$ and $\iota_{ri,ei}$ are the natural inclusions induced by $\iota_{T,r}$ and $\iota_{r,e}$, respectively. $\iota_{r,ri}$ is the inclusion map induced by taking a continuous degree to its downwards cone.

Note that in our language, paradoxically enough, continuous degrees are discrete degrees, because they are enumeration degrees.

It is natural to wonder about the relationship between $\mathscr{D}_r$, $\mathscr{D}_{ae}$, and $\mathscr{D}_a$ for two reasons: $\mathscr{D}_r$ has natural (proper) inclusion maps $\mathscr{D}_T \to \mathscr{D}_r \to \mathscr{D}_e$, and, ostensibly, elements of the Hilbert cube are the same objects that our ae- and a-degrees are degrees of.

A quick inspection shows that there cannot be an immediate relationship between $P \leq_r Q$ and $P \leq_a Q$. The precise value of the first term $Q(0)$ is entirely trivial to analog reducibility and wholly accessible to representation reducibility. Every a-degree contains oracles that are entirely rationally valued and entirely irrationally valued, but non-total continuous degrees contain only points with both rational and irrational entries.

Intuitively speaking this isn't completely surprising. Representation reducibility is perhaps best thought of as being about the computational content of topological properties of $[0,1]^\omega$ under the product metric, whereas it is probably more correct to think of analog reducibility in terms of the uniform structure of $[0,1]^\omega$ as a subspace of ℓ^∞.

That said there are still non-trivial relationships between $\mathscr{D}_r$ and $\mathscr{D}_a$. A somewhat superficial one is that we can characterize $\leq_a$ in terms of $\leq_r$: $P \leq_a Q$ if and only if for every $\varepsilon > 0$ there is a $\delta > 0$ such that for any $Q' \in [0,1]^\omega$ with $\|Q - Q'\| < \delta$, there is a $P' \in [0,1]^\omega$ with $\|P - P'\| \leq \varepsilon$ such that $P' \leq_r Q'$. This follows in a fairly shallow way from Theorem 1.18—in that given $Q \in [0,1]^\omega$ there are arbitrarily good rational uniform approximations of Q that are $\leq_r Q$—but it may be slightly more aesthetically pleasing in that it doesn't refer to rational valued functions, which were necessary in order to make the Turing reduction well-defined in this context without needing to talk about names of functions to $[0,1]$.

In this section we will prove a non-trivial relationship between $\mathscr{D}_r$ and $\mathscr{D}_a$. In particular we will show that the continuous degrees are characterized amongst the ae-degrees as precisely those degrees that have both a total $[0,1]$-valued representation and a positive $\{0,1\}$-valued representation, i.e. that $\mathscr{D}_r = \mathscr{D}_e \cap \mathscr{D}_a$, as subsets of $\mathscr{D}_{ae}$. The majority of the hard work was already done for us in [1]. First we need to show that $\mathscr{D}_r \subseteq \mathscr{D}_a$.

Definition 3.2. For any $P : [0,1] \to \omega$, let

$$P^\circ(\langle n,m,k\rangle) = ((mP(k) - n) \vee 0) \wedge 1.$$

Proposition 3.3. *For any $P, Q \in [0,1]^\omega$, $P \leq_r Q$ if and only if $P^\circ \leq_a Q^\circ$.*

Furthermore for any $P \in [0,1]^\omega$,

$$\iota_{e,ae} \circ \iota_{r,e} \circ \mathbf{d}_r(P) = \mathbf{d}_{ae}(P^\circ \oplus (1 - P^\circ)).$$

Proof. Given Theorem 1.18, we only need to show that given any name of Q, we can compute arbitrarily good uniform approximations of Q° and that given a good enough uniform approximation of Q° we can compute a name of Q.

Let f be a name of Q. Fix rational $\varepsilon > 0$. Using the name, for any n, m, and k, we can compute $Q(k)$ to an accuracy of $\frac{\varepsilon}{m}$, giving us an approximation of $Q^\circ(\langle n, m, mk \rangle)$ to within ε.

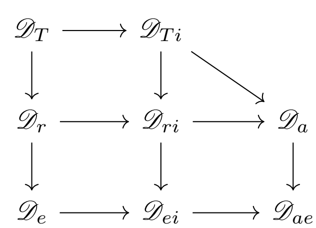
Let $Q' : \omega \to \mathbb{Q}$ be such that $\|Q - Q'\| < \frac{1}{2}$. By checking larger and larger values of m, we can find arbitrarily good approximations of $Q(k)$ for any k, so we can compute a name of Q.

For the furthermore part, first we just need to note that the $\{0,1\}$-valued positive oracle $P^\dagger = \{\langle r, n, 0 \rangle : r \in \mathbb{Q} \wedge r < P(n)\} \cup \{\langle r, n, 1 \rangle : r \in \mathbb{Q} \wedge r > P(n)\}$ clearly has $P^\circ \oplus (1 - P^\circ) \leq_{ae} P^\dagger$. For the other direction we just need a procedure that enumerates $P^\dagger$ given an approximation R satisfying $\|P^\circ - S\| < \frac{1}{3}$. Given such an approximation if we choose a k and an m and we look at the values of $S(\langle n, m, k \rangle)$ for $0 \leq n \leq m$, these will always have the property that there are $a \leq b$ with $0 \leq a \leq b \leq m$ such that for all $n < a$, $S(\langle n, m, k \rangle) > \frac{2}{3}$, and for all $n > b$, $S(\langle n, m, k \rangle) < \frac{1}{3}$. Given this information it's safe to enumerate any lower bound on $P(k)$ of the form $\frac{m-n}{m}$ for $n > b + 1$ and upper bounds of the form $\frac{m-n}{m}$ for $n < a - 1$, as well as any other rational upper and lower bounds on $P(k)$ that are consistent with those. Since we can do this for arbitrarily large m, this will eventually enumerate all rational lower and upper bounds on $P(k)$, so by Theorem 1.18 we have that $P^\dagger \leq_{ae} P^\circ \oplus (1 - P^\circ)$. □

Corollary 3.4. *As a subset of* $\mathscr{D}_e$ *(thought of as a subset of* $\mathscr{D}_{ae}$*),* $\mathscr{D}_r$ *is a subset of* $\mathscr{D}_a$*.*

Definition 3.5. Let $\iota_{ri,a} : \mathscr{D}_{ri} \to \mathscr{D}_a$ be the inclusion map induced by taking an r-ideal $\{P_i\}_{i<\omega}$ to $\bigoplus_{i<\omega} 2^{-i} P_i^\circ$.

Corollary 3.6. *The following diagram commutes.*



All inclusions preserve jumps, $\mathbf{0}$*, joins, and (where applicable)* ω*-joins.*

Note that in the proof of Proposition 3.3 we didn't actually need arbitrarily good uniform approximations of Q, we just needed a single good enough approximation. Such degrees have a sort of 'fault tolerance.' This raises a few questions.

Definition 3.7. An ae-degree (resp. a-degree) $\mathbf{a}$ is **fault tolerant** if for any Q with $\mathbf{a} = \mathbf{d}_{ae}(Q)$ (resp. $\mathbf{a} = \mathbf{d}_a(Q)$) there is an $\varepsilon > 0$ such that for any $Q' : \omega \to [0,1]$ with $\|Q - Q'\| < \varepsilon$, $Q \leq_{ae} Q'$ (resp. $Q \leq_a Q'$).

Note that this is a degree notion, not just an oracle notion. Any discrete degree is fault tolerant. Also, there are certainly non-fault tolerant *ae*- and *a*-degrees, namely any non-trivial ω-join. Fault tolerant degrees behave a little bit more like discrete degrees—for instance an *ae*- or *a*-degree can only have countably many fault tolerant predecessors—which motivates parts *(iii)* and *(iv)* of this question (the analogy to discrete degree would suggest that the answer to *(iii)* ought to be no).

Question 3.8. *Are there any ae- or a-degrees that are:*

(i) Neither fault tolerant nor a non-trivial ω-join?
(ii) Fault tolerant but not discrete?
(iii) Fault tolerant and a jump fixed point?
(iv) Not an ω-join of fault tolerant degrees?

Finally we just need to show the other direction of the inclusion.

Proposition 3.9. $\mathscr{D}_e \cap \mathscr{D}_a \subseteq \mathscr{D}_r$

Proof. Assume that $P : \omega \to [0, 1]$ and $X \subseteq \omega$ have that $\mathbf{d}_a(P) = \mathbf{d}_e(X)$, as *ae*-degrees. Unpacking definitions and using the fact that X is discrete, this implies that there is a rational $\varepsilon > 0$ small enough that

- the collection of $P' \in [0, 1]^\omega$ such that $\|P - P'\| \leq \varepsilon$ is a $\Pi_1\langle X\rangle$ class, F, as a subset of the Hilbert cube, and
- for any $P' \in F$, X is c.e. in any name of P'.

This is precisely codability, as defined in [1], so X has continuous degree by Theorem 1.3 of [1]. □

Corollary 3.10. *As subsets of $\mathscr{D}_{ae}$, $\mathscr{D}_r = \mathscr{D}_e \cap \mathscr{D}_a$.*

This raises another question.

Question 3.11. *What is the relationship between $\mathscr{D}_{ri}$ and $\mathscr{D}_{ei} \cap \mathscr{D}_a$?*

4. Generic Analog Degrees

The goal of this section is to show that all of the inclusions $\iota_{ri,a}$, $\iota_{ei,ae}$, and $\iota_{a,ae}$ are proper.

The simplest to exhibit non-total enumeration degrees are generic enumeration degrees with the property that they are not $\mathbf{0}_e$, but if they enumerate a set of the form $X \oplus \overline{X}$, then X is computable. Almost the same proof verbatim demonstrates that if $\mathbf{a}$ is a sufficiently generic discrete enumeration degree and $\mathbf{b}$ is an *a*-degree such that $\mathbf{b} \leq_{ae} \mathbf{a}$, then $\mathbf{b} = \mathbf{0}_a$, where generic is in the sense of co-meager in the Hilbert cube. To be more explicit:

Proposition 4.1. *For any sufficiently generic $Q : \omega \to [0, 1]$, if $P \oplus (1-P) \leq_{ae} Q$ for some $P : \omega \to [0, 1]$, then $P \equiv_a \mathbf{0}_a$.*

Proof. We will build a generic $[0, 1]$-valued oracle in stages. Let $\{\varphi_i(x, Q)\}_{i<\omega}$ be an enumeration of all of the restricted $\Sigma_1\langle Q\rangle$ formulas. At each stage we will have restricted ourselves to some topologically open cube $C_s = \{Q \in [0, 1]^\omega : (\forall i < k) r_i < Q(i) < s_i\}$ for some finite sequence (r_i, s_i). We will also require that the r_i and s_i be rational. Say that the open cube C_{s+1} is 'strongly contained' in C_s, written $C_{s+1} \subset\subset C_s$ if the topological closure of C_{s+1} is contained in C_s.

At stage $s + 1 = 2\langle i, k\rangle$, find an open cube $C_{s+1} \subset\subset C_s$ such that either

- for some n, for every $Q \in C_{s+1}$, $\varphi_i(2n, Q) + \varphi_i(2n+1, Q) > 1 + 2^{-k}$, or
- for some n, for every $Q \in C_{s+1}$, $\varphi_i(2n, Q) + \varphi_i(2n+1, Q) < 1 - 2^{-k}$,

if possible, otherwise, skip this step and let $C_{s+1} = C_s$.

At stage $s + 1 = 2n + 1$, find an open cube $C_{s+1} \subset\subset C_s$ such that the corresponding sequence has length at least n and for each i, $|s_i - r_i| < 2^{-n}$. (This is to ensure that we converge to an oracle.)

Note that since these strategies still work if we are playing against an opponent who is also shrinking the open cube, there is a co-meager set of Q with the properties guaranteed by this construction.

Given our actions on steps $s + 1 = 2n + 1$, we have that $\bigcap_{s<\omega} C_s$ is a singleton. Let Q be its only element. Assume that $P \oplus (1-P) \leq_{ae} Q$. This implies that there is a sequence of indices $\{i(k)\}_{k<\omega}$ such that $|(P \oplus (1-P))(n) - \varphi_{i(k)}(n, Q)| < 2^{-k}$ for every n and k. Therefore we have that for each $i(k)$, the construction of Q must have failed at stage $s + 1 = 2\langle i(k), k\rangle$.

This implies that for every open cube $C' \subset\subset C_s$, we have that

- for all n, there exists $R \in C'$ such that $\varphi_{i(k)}(2n, R) + \varphi_{i(k)}(2n+1, R) \leq 1 + 2^{-k}$, and
- for all n, there exists $R \in C'$ such that $\varphi_{i(k)}(2n, R) + \varphi_{i(k)}(2n+1, R) \geq 1 - 2^{-k}$.

The claim is that this implies that there is a uniformly lower semi-computable $P' : \omega \to [0,1]$ such that $\|P - P'\| \leq 2^{-k+1}$. Specifically, if R_s is the pointwise maximal oracle contained in $\overline{C}_s$ (which is a rational sequence that is eventually all 1s, and so computable), then we want $P'(n) = \varphi_{i(k)}(2n, R_s)$. By monotonicity of Σ_1 formulas, we have that $P'(n) \geq P(n) - 2^{-k+1}$ for all $n < \omega$. Now assume that $P'(n) > P(n) + 2^{-k+1}$ for some n. Then by uniform continuity of $\varphi_{i(k)}$ there is an open cube $C' \subset\subset C_s$ whose elements are close enough to R_s such that for any $R \in C'$, $\varphi_{i(k)}(2n, R) > P(n) + 2^{-k+1}$, but this contradicts the fact that the construction failed at stage s, so we must have that $P'(n) \leq P(n) + 2^{-k+1}$ as well.

Likewise, by symmetry, there is a uniformly upper semi-computable $P'' : \omega \to [0,1]$ such that $\|P - P''\| \leq 2^{-k+1}$. Hence, there is a computable $P''' : \omega \to [0,1]$ with $\|P - P'''\| < \frac{3}{2}2^{-k+1}$. Specifically, to compute $P'''(n)$, watch the enumerations of the lower bounds of $P'(n) - 2^{-k+1}$ and the upper bounds of $P''(n) + 2^{-k+1}$ until bounds that are within $3 \cdot 2^{-k+1}$ of each other are enumerated, and then let $P'''(n)$ be their average. This will necessarily be within $\frac{3}{2}2^{-k+1}$ of $P(n)$.

Since we can do this for any $k < \omega$, we have that P is approximated arbitrarily well by computable functions, so $P \equiv_a \mathbf{0}_a$, as required. □

In this section we will use the word 'metric' or 'metrically' to refer to things that are determined by the uniform metric in $[0,1]^\omega$ (e.g. a 'metrically closed ball') and we will use the word 'topological' or 'topologically' to refer to things that are determined by the product topology on $[0,1]^\omega$ (e.g. a 'topologically open set'). Also 'balls' will always be relative to the uniform metric (as we prefer to not even think of the product topology as arising from a metric) and 'generic' will always mean topologically co-meager.

We will also want this 'continuous' analog of the relatively trivial fact that a countable subset of Cantor space is meager.

Lemma 4.2. *If $A \subseteq [0,1]^\omega$ is metrically separable, then it is topologically meager.*

Proof. Let A_0 be a countable subset of A that is metrically dense in A.

Note that for any non-empty open (in the product topology) $U \subseteq [0,1]^\omega$, U is not covered by countably many closed balls of radius $\leq \frac{1}{6}$ (because any such U contains an uncountable set of points with pairwise distance 1). Also note that metrically closed balls are closed in the product topology. This implies that any countable union of closed balls of radius $\leq \frac{1}{6}$ is topologically meager.

So we have that $\bigcup_{a \in A_0} B_{\leq \frac{1}{6}}(a) \supseteq A$ is meager. Therefore A is meager as well. $\square$

The point being that sufficiently generic $[0,1]$-valued oracles have non-zero a and ae degree.

Proposition 4.3. *For any sufficiently generic $Q : \omega \to [0,1]$, if $P \leq_{ae} Q \oplus (1-Q)$ for some $P : \omega \to \{0,1\}$, then $P \equiv_e \mathbf{0}_e$.*

Proof. We will build a generic $[0,1]$-valued oracle in stages. Let $\{\varphi_i(x,Q)\}_{i<\omega}$ be an enumeration of all restricted $\Sigma_1(Q)$ formulas. Just like in the proof of Proposition 4.1, we will force with open cubes $C_s \subseteq [0,1]^\omega$. $\{G_k\}_{k<\omega}$ is a sequence of initially undefined subsets of $[0,1]^\omega$.

At stage $s+1 = 3i$, consider φ_i. For any n, the set of $Q \in C_s$ such that $\varphi_i(n,Q) > 0$ is topologically open, because whenever $\varphi_i(n,Q) > 0$, this only relies on some finite initial segment of Q and then by uniform continuity any Q' sufficiently close on that initial segment will also have $\varphi_i(n,Q) > 0$. So conversely, the set of $Q \in C_s$ such that $\varphi_i(n,Q) = 0$ is topologically closed (relative to C_s).

Find an n such that $F_n^s = \{Q \in C_s : \varphi_i(n,Q) = 0\}$ has non-empty interior but for which F_n^s is a proper subset of C_s. Let U be its interior (relative to C_s), which by assumption is non-empty. Find an $\varepsilon > 0$ such that for any P and R if $\|P - R\| < \varepsilon$, then $|\varphi(m,P) - \varphi(m,R)| < \frac{1}{2}$ for every m. Now for any set A, let $A^{<\varepsilon} := \{P \in C_s : (\exists R \in A)\, \|P - R\| < \varepsilon\}$ and consider $U^{<\varepsilon}$.

$U^{<\varepsilon}$ is topologically open. To see this note that for any family $\{A_i\}_{i\in I}$, $\left[\bigcup_{i\in I} A_i\right]^{<\varepsilon} = \bigcup_{i\in I} A_i^{<\varepsilon}$ and for an open cube C, $C^{<\varepsilon}$ is clearly open (it's just a large open cube), so since open cubes are a basis of the topology on $[0,1]^\omega$, for any open U, $U^{<\varepsilon}$ is open.

Now the claim is that $U^{<\varepsilon} \setminus F_n^s$ is non-empty. To see that this is true, note that U is not all of C_s and C_s is metrically connected, so $U^{<\varepsilon}$ must be a proper superset of U, otherwise U would be metrically clopen. For any set G if V is an open set that is a proper superset of the interior of G, $V \setminus G$ must be non-empty. Therefore we have that $U^{<\varepsilon} \setminus F_n^s$ is a non-empty open subset of C_s. Let C_{s+1} be some open cube such that $C_{s+1} \subset\subset U^{<\varepsilon} \setminus F_n^s$.

If no such n exists, find an increasing sequence $\{k_i\}_{i<\omega}$ such that

- $3k_i + 1 > s + 1$ for every i,
- no G_{k_i} has been defined yet, and
- the set of k such that G_k is still undefined is infinite.

Then set $G_{k_n} = F_n^s$ for each n where F_n^s is not all of C_s. If F_n^s is all of C_s, leave G_{k_n} undefined.

At stage $s+1 = 3k+1$, pass to $C_{s+1} \subset\subset C_s$ avoiding G_k, if it exists, otherwise let $C_{s+1} = C_s$. Avoiding is always possible since $G_k \cap C_s$ is nowhere dense and relatively closed (in C_s).

At stage $s+1 = 3k+2$, find $C_{s+1} \subset\subset C_s$ such that on the first k coordinates the width of the cube is $< 2^{-k}$. This is always possible.

Note that since these strategies still work if we are playing against an opponent who is also shrinking the open cube, there is a co-meager set of Q with the properties guaranteed by this construction.

By our actions on stages $s+1 = 3k+2$, $\bigcap_{s<\omega} C_s$ is a singleton. Let Q be its only element.

Note that if P is $\{0,1\}$-valued and $P \leq_{ae} Q \oplus (1-Q)$, then there must be a restricted $\Sigma_1(Q)$ formula witnessing this. This is because there is necessarily a restricted $\Sigma_1(Q)$ formula $\psi(n,Q)$ such that $|P(n) - \psi(n,Q)| < \frac{1}{3}$ for all n. Then we can take $((6(\psi(n,Q) - \frac{1}{2}) + 3) \vee 0) \wedge 1$ and this will be equal to $P(n)$.

Also note that if the construction succeeded on some stage $s+1 = 3i$, then $\varphi_i(n,Q)$ must fail to be $\{0,1\}$-valued. In particular, we have ensured that for some particular n, $0 < \varphi_i(n,Q) < \frac{1}{2}$.

So if $P(n) = \varphi_i(n,Q)$ is $\{0,1\}$-valued, the construction must have failed at stage $s+1 = 2i$. This means that for every n, either F_n^s is all of C_s or it has empty interior. For any n, if F_n^s is all of C_s, then clearly $P(n) = \varphi_i(n,Q) = 0$. Otherwise if F_n^s is not all of C_s, then by an action on some stage of the form $s+1 = 3k+2$, we forced that $P(n) = \varphi_i(n,Q) > 0$, so it must be the case that $P(n) = \varphi_i(n,Q) = 1$.

Given this we can computably enumerate P as follows: For each n, search for a finite $R \in ([0,1] \cap \mathbb{Q})^{<\omega}$ that is consistent with C_s (which is computable, since C_s has rational endpoints) such that $\varphi_i(n,R)$ outputs a lower bound > 0, in which case enumerate n. If $P(n) = 0$ this procedure will never find such an R, otherwise if $P(n) = 1$ this procedure will eventually enumerate n, so we have that P is c.e., as required. □

Now we can finally show that our inclusions are all proper.

Corollary 4.4. *For any sufficiently generic* $P \in [0,1]^\omega$, $\mathbf{d}_a(P) \in \mathscr{D}_a \setminus \mathscr{D}_{ri}$ *and* $\mathbf{d}_{ae}(P) \in \mathscr{D}_{ae} \setminus (\mathscr{D}_{ei} \cup \mathscr{D}_a)$.

Proof. By Lemma 4.2, $\mathbf{d}_a(P) \neq \mathbf{0}_a$ and $\mathbf{d}_{ae}(P) \neq \mathbf{0}_{ae}$. By Proposition 4.3, there is no non-c.e. $\{0,1\}$-valued X with $X \leq_{ae} P \oplus (1-P)$ and therefore also no such X with $X \leq_{ae} P$. This also implies that there is no incomputable $\{0,1\}$-valued Y such that $Y \leq_a P$. Since any non-zero element of $\mathscr{D}_{ri}$ is above some incomputable set (every non-zero continuous degree upper bounds a non-zero total degree), this implies that $\mathbf{d}_a(P)$ cannot be any non-zero element of $\mathscr{D}_{ri}$ either, so $\mathbf{d}_a(P) \in \mathscr{D}_a \setminus \mathscr{D}_{ri}$. Since no non-c.e. X can be $\leq_{ae} P$, we must have that $\mathbf{d}_a(P) \in \mathscr{D}_{ae} \setminus \mathscr{D}_{ei}$ as well.

Finally, by Proposition 4.1 we have that $\mathbf{d}_a(P) \in \mathscr{D}_{ae} \setminus \mathscr{D}_a$ as well, since it cannot be above any non-zero total oracle. So we have $\mathbf{d}_{ae}(P) \in \mathscr{D}_{ae} \setminus (\mathscr{D}_{ei} \cup \mathscr{D}_a)$, as required. □

Of course, it is easy to ask followup questions to this, such as

Question 4.5. *If* $Q \in [0,1]^\omega$ *is sufficiently random, does it follow that if* $P \leq_{ae} Q \oplus (1-Q)$ *and* P *is* $\{0,1\}$*-valued, then* $P \equiv_e \mathbf{0}_e$*?*

5. More Restrictive Forms of Analog Reduction

There are at least three reasonable more restrictive versions of $\leq_a$ and $\leq_{ae}$. It is unlikely that they are equivalent. For the following definition, note that if a discrete set $X \subseteq \omega$ satisfies $X \leq_a P$ for some $P : \omega \to [0,1]$, then there is a restricted formula witnessing this, so X is computable from P in the traditional sense (although note that the converse fails).

Definition 5.1. For any $X \subseteq \omega$, a $\Delta_0(P)$ formula $\varphi(x, P)$ (where P is a variable, rather than a particular oracle) is X-computable if there is an X-computable map from names for oracles $P : \omega \to [0, 1]$ to $n \mapsto \varphi(n, P)$.

For any $Q : \omega \to [0, 1]$, a $\Delta_0(P)$ formula $\varphi(x, P)$ is ***Q*-computable** if there is some $X \subseteq \omega$ with $X \leq_a Q$ such that φ is X-computable. $\varphi(x, P)$ is **positively *Q*-computable** if there is some $X \subseteq \omega$ with $X \leq_{ae} Q$ such that φ is X-computable.

We say that a $\Sigma_1(P)$ formula is X-computable (resp. Q-computable, positively Q-computable) if the underlying $\Delta_0(P)$ formula is X-computable (resp. Q-computable, positively Q-computable).

Definition 5.2. For any $P, Q : \omega \to [0, 1]$ we define the following.

- We say that P is **strongly computably analog enumeration reducible to *Q***, written $P \leq_{\mathrm{s}ae} Q$ if there is some computable $\Sigma_1\langle Q\rangle$ formula $\varphi(x)$ such that for all $n < \omega$, $\varphi(n) = P(n)$.
- We say that P is **weakly computably analog enumeration reducible to *Q***, written $P \leq_{\mathrm{w}ae} Q$ if there is some positively Q-computable $\Sigma_1\langle Q\rangle$ formula $\varphi(x)$ such that for all $n < \omega$, $\varphi(n) = P(n)$.

For $* \in \{\mathrm{s}, \mathrm{w}\}$, we say that $P \leq_{*a} Q$ if the analogous condition holds for some $\Sigma_1(Q)$ formula (in particular for weakly computably analog reducible, we only require that the formula be Q-computable). We also define $\mathscr{D}_{*ae}$ and $\mathscr{D}_{*a}$ in the obvious way.

Note that by the comments at the beginning of this section, $\leq_{\mathrm{w}ae}$ and $\leq_{\mathrm{wa}}$ have the countable predecessor property. The strong reductions clearly have the countable predecessor property.

One could also require that the witnessing formula be restricted, so that it would be a truly finitary object, but this notion is slightly arbitrary, depending on the particular definition of restricted formula, and seems very poorly behaved. For example, $P(n) = \frac{1}{1+n}$ has non-trivial degree under such a reduction.

Proposition 5.3. *For any $P, Q : \omega \to [0, 1]$,*

- *if $P \leq_{\mathrm{s}ae} Q$, then $P \leq_{\mathrm{w}ae} Q$, and*
- *if $P \leq_{\mathrm{w}ae} Q$, then $P \leq_{ae} Q$.*

$P \leq_{ae} Q$ does not in general imply $P \leq_{\mathrm{w}ae} Q$. The same statements are true if we replace ae with a.

Proof. The implications for both the ae and a versions of the proposition are trivial. The fact that $P \leq_a Q$ does not generally imply $P \leq_{\mathrm{wc}-a} Q$ follows from cardinality considerations. □

Reversal of the first implication is unclear at the moment.

Question 5.4. *Does $P \leq_{\mathrm{s}ae} Q$ imply $P \leq_{\mathrm{w}ae} Q$? Does $P \leq_{\mathrm{sa}} Q$ imply $P \leq_{\mathrm{wa}} Q$?*

Many of the degree theoretic results in this paper still hold for these more restrictive notions, such as

- the inclusion of $\mathscr{D}_{*a}$ into $\mathscr{D}_{*ae}$ (Corollary 1.16),
- the inclusions of $\mathscr{D}_T$ and $\mathscr{D}_e$ into $\mathscr{D}_{*a}$ and $\mathscr{D}_{*ae}$, respectively (Proposition 2.2 for $\mathscr{D}_T$ and $\mathscr{D}_e$, rather than $\mathscr{D}_{Ti}$ and $\mathscr{D}_{ei}$),

- the inclusion of $\mathscr{D}_r$ into $\mathscr{D}_{*a}$ (Proposition 3.3 and Corollary 3.4, note that the relevant direction of Theorem 1.18 only uses restricted formulas),
- the characterization of $\mathscr{D}_r$ as $\mathscr{D}_{*a} \cap \mathscr{D}_e$ (Proposition 3.9), and
- the non-discreteness of sufficiently generic analog degrees (Corollary 4.4).

The other results of this paper are less clear in this context.

Question 5.5. *To what extent can Theorem 1.18 be recovered for $\leq_{*a}$ and $\leq_{*ae}$ with $* \in \{\mathrm{sc}, \mathrm{wc}\}$?*

Question 5.6. *Is there a natural embedding of $\mathscr{D}_{*a}$ and $\mathscr{D}_{*ae}$ with $* \in \{\mathrm{sc}, \mathrm{wc}\}$ into the Muchnik degrees (analogously to Corollary 1.20)?*

6. The Hereditarily Compact Superstructure

Definition 6.1. Let $\mathfrak{M}$ be a single-sorted metric structure. The **Hereditarily Compact Superstructure of $\mathfrak{M}$**, written $\mathrm{HK}(\mathfrak{M})$, is a two-sorted metric structure whose first sort, A, is $\mathfrak{M}$ and whose second sort, S, is the metric closure of the collection of hereditarily finite sets with atoms from $\mathfrak{M}$ (i.e. $\mathrm{HF}(\mathfrak{M}) \setminus \mathfrak{M}$) under the following recursively defined metric:

$$d^{\mathrm{HF}(\mathfrak{M})}(A, B) = d_H^{\mathfrak{M}}(A \cap \mathfrak{M}, B \cap \mathfrak{M}) \vee d_H^{\mathrm{HF}(\mathfrak{M})}(A \setminus \mathfrak{M}, B \setminus \mathfrak{M}),$$

where d_H is the Hausdorff metric on sets. For a non-empty set $X \subseteq Y \in \{\mathfrak{M}, \mathrm{HF}(\mathfrak{M})\}$, we take $d_H^Y(X, \varnothing)$ to be the syntactic diameter of $\mathfrak{M}$ (i.e. the diameter as specified by the signature of the structure $\mathfrak{M}$, rather than its actual diameter as a metric space).

The second sort is taken to have the same syntactic diameter as the syntactic diameter of $\mathfrak{M}$, and we have a pair of new binary predicates, one on $A \times S$ and the other on $S \times S$, both written with the symbol E. For $x \in \mathrm{HK}(\mathfrak{M})$ and $y \in \mathrm{HF}(\mathfrak{M}) \setminus \mathfrak{M}$, we take

$$E^{\mathrm{HK}(\mathfrak{M})}(x, y) = \inf_{z \in y} d^{\mathrm{HK}(\mathfrak{M})}(x, z),$$

where again this infimum is taken to be the syntactic diameter of $\mathfrak{M}$ when the relevant infimum is empty.

If the signature of $\mathfrak{M}$ is $\mathcal{L}$, then we let $\mathcal{L}_{\mathrm{HK}}$ be the signature of $\mathrm{HK}(\mathfrak{M})$. This only depends on the choice of $\mathcal{L}$.

In a structure of the form $\mathrm{HK}(\mathfrak{M})$, a **pure set** is an element of HF as a substructure of $\mathrm{HK}(\mathfrak{M})$, i.e. the smallest class containing $\varnothing$ and closed under the formation of finite sets.

It is not hard to see that E is 2-Lipschitz in its arguments, or more precisely that $|E(x, y) - E(x', y')| \leq d(x, x') + d(y, y')$, so it extends uniquely to all of $\mathrm{HK}(\mathfrak{M})$. It is also not hard to see that for any fixed a, $E(x, a)$ is a distance predicate for a definable set (or more accurately a pair of distance predicates for a pair of definable sets) and in particular when $a \in \mathrm{HF}(\mathfrak{M}) \setminus \mathfrak{M}$, then $E(x, a)$ is the distance predicate for the set a. As such we will freely think of elements of $\mathrm{HK}(\mathfrak{M})$ as the sets they represent via the predicate E. Furthermore we will freely refer to the 'rank' of an element of $\mathrm{HK}(\mathfrak{M}) \setminus \mathfrak{M}$, which is well-defined, since elements of $\mathrm{HF}(\mathfrak{M})$ of distinct rank have maximal distance, so the rank function extends continuously to all of $\mathrm{HK}(\mathfrak{M})$.

Lemma 6.2. *A set $X \subseteq \mathrm{HK}(\mathfrak{M})$ corresponds to an element of $\mathrm{HF}(\mathfrak{M})$ if and only if it is a metrically compact subset of $\mathrm{HK}(\mathfrak{M})$.*

Proof. Assume that X corresponds to an element a of $\mathrm{HK}(\mathfrak{M})$. By definition this implies that there is a sequence $\{a_i\}_{i<\omega}$ of elements of $\mathrm{HF}(\mathfrak{M})$ limiting to a. This implies that X is totally bounded. Since it is also closed, it is metrically compact.

Let X be a metrically compact subset of $\mathrm{HK}(\mathfrak{M})$. Let $\{X_i\}_{i<\omega}$ be a sequence of subsets of $\mathrm{HF}(\mathfrak{M})$ limiting to X in the Hausdorff metric (this is possible because $\mathrm{HF}(\mathfrak{M})$ is dense in $\mathrm{HK}(\mathfrak{M})$. Furthermore, since X is metrically compact we may take the X_i to be individually finite. $\{X_i\}$ now corresponds to a sequence of elements of $\mathrm{HF}(\mathfrak{M})$ that is metrically convergent. Let $a \in \mathrm{HK}(\mathfrak{M})$ be its limit. By construction a corresponds precisely to the set X. □

Definition 6.3. For any single-sorted signature $\mathcal{L}$ and $\mathcal{L}_{\mathrm{HK}}$-formula φ that obeys the moduli of uniform continuity α in the variable x, we define the following **bound quantifiers**:

- $\sup_{x\in y} \varphi = \sup_x \varphi - \alpha(E(x,y))$
- $\inf_{x\in y} \varphi = \inf_x \varphi + \alpha(E(x,y))$

where y is a variable in the set sort of $\mathcal{L}_{\mathrm{HK}}$.

Lemma 6.4. *For structures of the form $\mathrm{HK}(\mathfrak{M})$, bound quantifiers are semantically accurate, i.e. $\mathrm{HK}(\mathfrak{M}) \models \sup_{x\in a} \varphi(x) > r$ if and only if $\sup\{\varphi^{\mathfrak{M}}(b) : E^{\mathfrak{M}}(b,a) = 0\} > r$, and likewise for $\inf$.*

In particular, the value of formulas involving bound quantifiers does not depend on the particular choice of the modulus α (as long as it is actually a modulus of uniform continuity for the formula in question).

Proof. This follows from the fact that in structures of the form $\mathrm{HK}(\mathfrak{M})$, $E(x,a)$ is the distance predicate of a definable set for any a. □

The definitions of Δ_0 and Σ_1 formulas in the context of structures of the form $\mathrm{HK}(\mathfrak{M})$ is analogous to Definition 1.8. Likewise for the definition of restricted Δ_0 and Σ_1 formulas.

Lemma 6.5. *If φ is a Δ_0 formula then for any $\varepsilon > 0$ there is a restricted Δ_0 formula ψ such that for any $\mathfrak{M}$ (in the appropriate signature) and $\bar{a} \in \mathrm{HK}(\mathfrak{M})$, $|\varphi^{\mathrm{HK}(\mathfrak{M})}(\bar{a}) - \psi^{\mathrm{HK}(\mathfrak{M})}(\bar{a})| < \varepsilon$.*

Proof. Essentially the same as the proof of Corollary 1.12. □

6.1. Σ_1 **Formulas on** $\mathrm{HK}(\mathfrak{M})$**.** Now we will give a r.i.c.e.-style characterization of Σ_1 formulas on structures of the form $\mathrm{HK}(\mathfrak{M})$. This is of course a generalization of Theorem 1.18.

In the following theorem, we only need the additional relation R for the results in Subsection 6.2.

Theorem 6.6. *For any modulus of uniform continuity α, separable single-sorted structure $\mathfrak{M}$ in a finite signature $\mathcal{L} = \{Q_0, Q_1, \dots, f_0, f_1, \dots\}$, and relations $P, R : \mathrm{HK}(\mathfrak{M}) \to [0,1]$, with R α-uniformly continuous, the following are equivalent:*

(i) P is $\Sigma_1(R)$ definable with parameters in $\mathrm{HK}(\mathfrak{M})$.
(ii) For any $\varepsilon > 0$ there is a $\delta > 0$ such that any approximate presentation of $(\mathrm{HK}(\mathfrak{M}), R)$ to within accuracy δ enumerates an approximation of P to within accuracy ε. More formally, for every $\varepsilon > 0$ there is a $\delta > 0$ such

that for any $A = \{a_i\}_{i<\omega}$, a sequence of elements of $\mathfrak{M}$, with $\mathfrak{M} \subseteq A^{<\delta} := \{x : (\exists y \in A) d(x,y) < \delta\}$, and any sequence $\{q_j\}_{j<\omega}$ of rational numbers, if:

- *$|q_{\langle 0,i,j\rangle} - d^{\mathfrak{M}}(a_i, a_j)| < \delta$, for each $i, j < \omega$,*
- *$|q_{\langle 1,i,\bar{k}\rangle} - Q_i^{\mathfrak{M}}(a_{k_0}, a_{k_1}, \dots)| < \delta$, for each i such that $Q_i \in \mathcal{L}$ and each tuple $\bar{k}$ of the appropriate length,*
- *$|q_{\langle 2,i,\bar{k}\rangle} - d^{\mathfrak{M}}(a_{k_0}, f_i^{\mathfrak{M}}(a_{k_1}, a_{k_2}, \dots))| < \delta$, for each i such that $f_i \in \mathcal{L}$ and each tuple $\bar{k}$ for the appropriate length, and*
- *$|q_{\langle 3,\{b\}\rangle} - R(b^{\mathrm{HK}(\mathfrak{M})})| < \delta$, for each $b \in \mathrm{HF}(\omega)$ where $\{b\}$ is some encoding of the elements of $\mathrm{HF}(\omega)$ as natural numbers and $b^{\mathrm{HK}(\mathfrak{M})}$ is understood to be the corresponding element of $\mathrm{HF}(A) \subseteq \mathrm{HK}(\mathfrak{M})$,*

where $\langle -, -, \dots \rangle$ is understood to be an encoding of finite sequences of natural numbers as natural numbers, then there is a function $P' : \mathrm{HF}(A) \to [0,1]$ that is uniformly lower semi-computable in $\{q_j\}$ such that for every $a \in \mathrm{HF}(A)$, $|P(a) - P'(a)| \le \varepsilon$.

Proof. *(i)⇒(ii)*: Assume that $P(y)$ is $\Sigma_1(R)$ definable by a formula of the form

$$\sup_{x_0} \sup_{x_1} \dots \varphi(\bar{x}, \bar{a}, y),$$

with φ a $\Delta_0(R)$ formula. Fix $\varepsilon > 0$ and find a restricted $\Delta_0(R)$ formula ψ such that for any structure $\mathfrak{N}$ in the same signature as $\mathfrak{M}$, any $\bar{b}, \bar{c}, e \in \mathfrak{N}$, and any α-continuous $R' : \mathrm{HK}(\mathfrak{N}) \to [0,1]$,

$$|\varphi^{(\mathrm{HK}(\mathfrak{N}),R')}(\bar{b}, \bar{c}, e) - \psi^{(\mathrm{HK}(\mathfrak{N}),R')}(\bar{b}, \bar{c}, e)| < \frac{\varepsilon}{3}.$$

Now let $x_0, x_1, \dots, x_{k-1}$ be the set of free variables of the form x_i appearing in ψ (which is finite even if φ has infinitely many free variables, since ψ is restricted) and let $\bar{a} = a_0, a_1, \dots, a_{\ell-1}$ be the set of parameters corresponding to variables appearing in ψ (also finite). Now find $\gamma > 0$ small enough that if $d(\bar{b}\bar{c}e, \bar{b}'\bar{c}'e') < \gamma$ then $|\psi^{(\mathrm{HK}(\mathfrak{M}),R)}(\bar{b}, \bar{c}, e) - \psi^{(\mathrm{HK}(\mathfrak{M}),R)}(\bar{b}', \bar{c}', e')| < \frac{\varepsilon}{3}$.[3] Finally find $\delta > 0$ small enough that $\delta < \gamma$ and for any $\bar{b}, \bar{c}, e \in \mathrm{HK}(\mathfrak{M})$, if $\psi'(\bar{b}, \bar{c}, e)$ is the result of modifying each value of an atomic predicate in $\psi^{(\mathrm{HK}(\mathfrak{M}),R)}(\bar{b}, \bar{c}, e)$ by at most δ, then $|\psi'(\bar{b}, \bar{c}, e) - \psi^{(\mathrm{HK}(\mathfrak{M}),R)}(\bar{b}, \bar{c}, e)| < \frac{\varepsilon}{3}$ (in particular it is sufficient to choose $\delta < \frac{\varepsilon}{3}\left[n \cdot \prod\{|r| \vee 1 : r \text{ a rational number appearing in } \psi\}\right]^{-1}$, where n is the number of atomic sub-formulas in ψ and the product is computed with multiplicity).

Now let $A = \{a_i\}_{i<\omega}$ be a sequence of elements of $\mathfrak{M}$ such that $\mathfrak{M} \subseteq A^{<\delta}$ and let $\{q_j\}_{j<\omega}$ be a sequence of rational numbers such that the approximation property in the statement of the theorem holds. Let $\bar{a}'$ be a tuple of elements of A such that for each $i < |\bar{a}|$, $d^{\mathfrak{M}}(a_i, a_i') < \delta < \gamma$. Now by construction for any $c \in A$, we have that $|\psi^{(\mathrm{HK}(\mathfrak{M}),R)}(\bar{b}, \bar{a}, c) - \psi^{(\mathrm{HK}(\mathfrak{M}),R)}(\bar{b}', \bar{a}', c)| < \frac{\varepsilon}{3}$.

Claim: $\mathrm{HF}(A)$ as a subset of $\mathrm{HK}(\mathfrak{M})$ has the property that $\mathrm{HK}(\mathfrak{M}) \subseteq \mathrm{HF}(A)^{<\delta}$.

Proof of claim: We clearly only need to focus on sets. Also note that $\mathrm{HF} \subseteq \mathrm{HF}(A)$, so any hereditarily pure set is covered. If δ is greater than the syntactic diameter of $\mathcal{L}$, then we are done, so assume that δ is $\le$ the syntactic diameter of $\mathcal{L}$. We proceed by induction on rank. Assume that we've shown that for any $a \in \mathrm{HK}(\mathfrak{M}) \setminus \mathfrak{M}$ of rank $< n$ that $d(a, \mathrm{HF}(A)) < \delta$. Let $a \in \mathrm{HK}(\mathfrak{M})$ have rank

[3] This is the only place where we use that R is α-uniformly continuous. Note that we don't actually need that α be computable.

n. By the induction hypothesis, a as a set of elements of $\mathrm{HK}(\mathfrak{M})$ is covered by $\mathrm{HF}(A)^{<\delta}$. Since δ is less than or equal to the syntactic diameter of $\mathcal{L}$, it must actually be covered by elements of rank $< n$. Since a is a compact set there is a finite set b of elements of $\mathrm{HF}(A)$ of rank $< n$ such that $a \subseteq b^{<\delta}$ and such that $b \subseteq a^{<\delta}$. Again by compactness, this implies that there is a $\sigma < \delta$ such that the same is true, so $d^{\mathrm{HK}(\mathfrak{M})}(a,b) \leq \sigma < \delta$.

So by induction we have that $\mathrm{HK}(\mathfrak{M}) \subseteq \mathrm{HF}(A)^{<\delta}$. $\square_{\text{claim}}$

Fix $c \in \mathrm{HF}(A) \subseteq \mathrm{HK}(\mathfrak{M})$ and assume that $\mathrm{HK}(\mathfrak{M}) \models \sup_{x_0} \sup_{x_1} \dots \varphi(\bar{x}, \bar{a}, c) > r$ (i.e. $P(c) > r$). This implies that there exists a sequence $\bar{b}$ of elements of $\mathrm{HK}(\mathfrak{M})$ such that $\varphi^{(\mathrm{HK}(\mathfrak{M}),R)}(\bar{b}, \bar{a}, c) > r$. Let $\bar{b}'$ be a sequence of elements of $\mathrm{HF}(A) \subseteq \mathrm{HK}(\mathfrak{M})$ such that for each i, $d^{\mathrm{HK}(\mathfrak{M})}(b_i, b_i') < \delta$ (this exists by the claim). So now by construction we have that

$$\begin{aligned}
|\varphi^{(\mathrm{HK}(\mathfrak{M}),R)}(\bar{b},\bar{a},c) - \psi^{(\mathrm{HF}(A),q_j)}(\bar{b}',\bar{a}',c)| &\leq |\varphi^{(\mathrm{HK}(\mathfrak{M}),R)}(\bar{b},\bar{a},c) - \psi^{(\mathrm{HK}(\mathfrak{M}),R)}(\bar{b},\bar{a},c)| \\
&\quad + |\psi^{(\mathrm{HK}(\mathfrak{M}),R)}(\bar{b},\bar{a},c) - \psi^{(\mathrm{HK}(\mathfrak{M}),R)}(\bar{b}',\bar{a}',c)| \\
&\quad + |\psi^{(\mathrm{HK}(\mathfrak{M}),R)}(\bar{b}',\bar{a}',c) - \psi^{(\mathrm{HF}(A),q_j)}(\bar{b}',\bar{a}',c)| \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon
\end{aligned}$$

where $\psi^{(\mathrm{HF}(A),q_j)}(\bar{b}', \bar{a}', c)$ is $\psi(\bar{b}', \bar{a}', c)$ evaluated using the values in $\{q_i\}$ in the appropriate way (note that in particular the function $(\bar{b}', c) \mapsto \psi^{(\mathrm{HF}(A),q_j)}(\bar{b}', \bar{a}', c)$ is computable in $\{q_j\}$.

This implies that $\sup\{\psi^{(\mathrm{HF}(A),q_j)}(\bar{x}, \bar{a}', c) : \bar{x} \in A\} \geq r - \varepsilon$, so in particular $\sup\{\psi^{(\mathrm{HF}(A),q_j)}(\bar{x}, \bar{a}', c) : \bar{x} \in \mathrm{HF}(A)\} \geq P(c) - \varepsilon$. A very similar argument gives us that $P(c) \geq \sup\{\psi^{(\mathrm{HF}(A),q_j)}(\bar{x}, \bar{a}', c) : \bar{x} \in \mathrm{HF}(A)\} - \varepsilon$, so if we set $P'(c) = \sup\{\psi^{(\mathrm{HF}(A),q_j)}(\bar{x}, \bar{a}', c) : \bar{x} \in \mathrm{HF}(A)\}$, we have that $|P(c) - P'(c)| < \varepsilon$ for all $c \in \mathrm{HF}(A)$. The function $i \mapsto P'(a_i)$ is lower semi-computable in $\{q_j\}$, as required.

(ii)$\Leftarrow$*(i)*: Assume that *(ii)* holds, fix $\varepsilon > 0$ and find $\delta > 0$ such that the statement holds.

Let $\{m_i\}_{i<\omega}$ be a countable dense subset of $\mathfrak{M}$. We're going to build a generic A in stages. At each stage our specification will be a finite list $\{a_i\}_{i<\ell_s}$ and a finite list $\{q_{\langle n,i,\bar{k}\rangle}\}_{n<4,i,\bar{k}<\ell_s}$. We'll call the specification at stage s, (A_s, Q_s), where A_s is the a's chosen so far and Q_s is the q's chosen so far.

On stage $s+1 = 2e$, find $(A_{s+1}, Q_{s+1}) \succeq (A_s, Q_s)$ such that all new q are correct to within $< 2^{-s-1}\delta$ and such that either

- there is an n such that $\Phi_e^{(A_{s+1},Q_{s+1})}(n)$ halts and enumerates a lower bound r for $P'(b)$ for some $b \in \mathrm{HF}(A_{s+1})$ (where ℓ_{s+1} is the length of A_{s+1}) such that $r > P(b) + 2\varepsilon$, or
- there is a $b \in \mathrm{HF}(A_{s+1})$ such that for every $(A', Q') \succ (A_{s+1}, Q_{s+1})$ (with q that are correct to within $< 2^{-s-2}$), for every m, $\Phi^{(A',Q')}(m)$ fails to halt and enumerate a lower bound r for $P'(b)$ with $r \geq P(b) - 2\varepsilon$.

If this is not possible, stop the construction as it has failed.

On stage $s+1 = 2i+1$, find a point $a \in \mathfrak{M}$ such that $d^{\mathfrak{M}}(a, m_i) < 2^{-s-1}\delta$ and choose values for the new q that are correct to within $< 2^{-s-1}\delta$. Append these to get A_{s+1} and Q_{s+1}.

Assume that the construction succeeds. Then by construction we have that any P' that is computable from (A, Q) has $\|P - P'\| \geq 2\varepsilon > \varepsilon$, which contradicts our assumption. Hence the construction must fail at some stage $s+1 = 2e$. This means that for any $(A', Q') \succ (A_s, Q_s)$ consistent with the specifications to within $< 2^{-s-1}\delta$,

- for every n if $\Phi_e^{(A',Q')}(n)$ halts and enumerates a lower bound for r for $P'(a)$ for some $a \in \mathrm{HF}(A')$, then $r \leq P(a) + 2\varepsilon$, and
- for any $a \in \mathrm{HF}(A')$ there is an $(A'', Q'') \succ (A', Q')$ such that for some m, $\Phi^{(A'',Q'')}$ halts and enumerates a lower bound r for $P'(a)$ with $r \geq P(a) - 2\varepsilon$.

Now, just as in the proof of Theorem 1.18 part *(i)*, we can find a discretization scale 2^{-N} small enough relative to $2^{-s-1}\delta$ and ε and produce a restricted $\Sigma_1(R)$ formula that approximates P to within $< 2\varepsilon + 2^{-N} < 3\varepsilon$.

Since we can do this for arbitrarily small ε, and since the analog of Lemma 1.13 holds for structures of the form $(\mathrm{HK}(\mathfrak{M}), R)$, we get that P has a $\Sigma_1(R)$ definition as well. □

6.2. **Characterizing a More Traditional Notion.** It is also possible to give a characterization of the more traditional notion of r.i.c.e. in a metric structure—namely those subsets of ω which are computably enumerable from any presentation of a countable dense sub-structure—in terms of $\mathrm{HK}(\mathfrak{M})$. As with the embedding of the continuous degrees into the analog degrees, the key is to present the relevant information at arbitrarily high precisions simultaneously.

An issue we run into is that we need to add a special predicate to $\mathrm{HK}(\mathfrak{M})$ to manage this information. This is analogous to how computability from structures in infinite signatures is sometimes defined with a single predicate uniformly encoding all predicate and function symbols, indexed by naturals (see II.4.1 in [9]).

For simplicity we will only prove this for $\mathfrak{M}$ a metric space. Note that by [7], any metric structure in a countable (computable) signature can be encoded as a metric space in a first-order, uniformly computable way. We will also assume that the syntactic diameter of the signature is 1.

Definition 6.7. For any metric space $\mathfrak{M}$ in the empty signature with syntactic diameter 1, the structure $\mathfrak{M}^*$ is a metric structure in a signature, with a new 1-Lipschitz unary function symbol f, with underlying set $M \times \omega$, and a metric defined by $d((x,n),(y,m)) = 2$ if $n \neq m$ and $d((x,n),(y,n)) = 2^n d(x,y) \wedge 1$, and f interpreted as $f((x,n)) = (x, n-1)$ if $n > 0$ and $f((x,0)) = (x,0)$.

Definition 6.8. For any metric space $\mathfrak{M}$ in the empty signature with syntactic diameter 1, the **modified hereditarily compact superstructure of $\mathfrak{M}$**, written $\mathrm{HK}^*(\mathfrak{M})$, is $\mathrm{HK}(\mathfrak{M}^*)$ with an additional 2-Lipschitz 4-ary predicate $S(x,y,n,m)$, with x and y variables in the atom sort and n and m variables in the set sort, such that

- $S^{\mathrm{HK}^*(\mathfrak{M})}((x_0,k_0),(x_1,k_1),n,m) = -1$ if either $k_0 < m$, $k_1 < m$, or either of x or y is not a natural number, and
- $S^{\mathrm{HK}^*(\mathfrak{M})}((x_0,k_0),(x_1,k_1),n,m) = [(2^m d^{\mathfrak{M}^*}((x_0,k_0),(x_1,k_1)) - n) \vee 0] \wedge 1$ otherwise.

Note that the first bullet point's restriction in Definition 6.8 ensures that M is actually 2-Lipschitz.

Proposition 6.9. *For any separable metric space $\mathfrak{M}$ and any $X \subseteq \omega$, the following are equivalent.*

(i) X is Σ_1 definable with parameters in $\mathrm{HK}^*(\mathfrak{M})$.

(ii) X is c.e. in any presentation of a countable dense subspace of $\mathfrak{M}$.

Proof. *(i)⇒ (ii)*: Assume that X is Σ_1 definable with parameters in $\mathrm{HK}^*(\mathfrak{M})$. Since X is a discrete set, there is a restricted Σ_1 formula $\varphi(x, \bar{b})$ with $\bar{b} \in \mathrm{HK}^*(\mathfrak{M})$ such that $\varphi^{\mathrm{HK}^*(\mathfrak{M})}(a)$ is in $\{0, 1\}$ and is 1 if and only if $a \in X$ (coded as a pure set). We may assume that $\bar{b}$ is in $\mathfrak{M}$ (as a sub-structure of $\mathfrak{M}^*$). Let $\mathfrak{M}_0$ be a dense sub-structure of $\mathfrak{M}$. We can find elements $\bar{b}_0 \in \mathfrak{M}_0$ close enough to $\bar{b}$ such that $2\varphi(x, \bar{b}_0) \wedge 1$ is still $\{0, 1\}$-valued and defines X, so we may assume that $\bar{b}$ is actually in $\mathfrak{M}_0$.

Given a presentation of $\mathfrak{M}_0$, it is clearly possible to (uniformly) compute a presentation of $\mathfrak{M}_0^*$ and then therefore also $\mathrm{HF}^*(\mathfrak{M}_0)$ (where $\mathrm{HF}^*(\mathfrak{A})$ is the restriction of $\mathrm{HK}^*(\mathfrak{A})$ to $\mathrm{HF}(\mathfrak{A})$). Let $\varphi(x, \bar{b})$ be of the form $\sup_y \psi(x, y, \bar{b})$. We know that a given natural number n is in X if and only if we can find $a \in \mathrm{HF}(\mathfrak{M}_0)$ such that $\mathrm{HF}^*(\mathfrak{M}_0) \models \psi(n, a, \bar{b}) > \frac{1}{2}$, so we can enumerate X from our presentation of $\mathrm{HF}^*(\mathfrak{M}_0)$ and therefore from our presentation of $\mathfrak{M}_0$.

(ii) ⇒ (i): Assume that X is c.e. in any presentation of a countable dense sub-structure of $\mathfrak{M}$. We need to show that given a sufficiently good approximate presentation of $\mathrm{HK}^*(\mathfrak{M})$ in the sense of part *(ii)* of Theorem 6.6 we can compute a presentation of a countable dense sub-structure of $\mathfrak{M}$.

Let $A = \{a_i\}_{i<\omega}$ be a sequence of elements of $\mathfrak{M}^*$ and let $\{q_i\}_{i<\omega}$, $\{r_i\}_{i<\omega}$, and $\{s_i\}_{i<\omega}$ be sequences of rational numbers such that for each $i, j < \omega$, $|q_{\langle i,j\rangle} - d(a_i, a_j)| < \frac{1}{4}$ and $|r_{\langle i,j\rangle} - d^{\mathfrak{M}^*}(a_i, f(a_j))| < \frac{1}{4}$ and for each $i, j, n, m < \omega$, $|s_{\langle i,j,n,m\rangle} - S^{\mathrm{HK}^*(\mathfrak{M})}(a_i, a_j, n, m)| < \frac{1}{4}$.

Associate to each a_i a sequence of indices $\{k_i(n)\}_{n<\omega}$ where $k_i(0) = i$ and for each $n < \omega$, $k_i(n+1)$ is the smallest k such that $r_{\langle k_i(n),k\rangle} < \frac{1}{4}$. In other words, we're grabbing the first thing that looks approximately like the pre-image of $a_{k_i(n)}$ under f. Such a k will always be found.

Let $f^\infty(c)$ be the eventual fixed point of iterating f on $c \in \mathfrak{M}^*$, regarded as an element of $\mathfrak{M}$. For each $i < \omega$, consider the sequence $\{f^\infty(a_{j_i(n)})\}_{n<\omega}$. By construction, we have that $d(f^\infty(a_{k_i(n)}), f^\infty(a_{k_i(n+1)}) < \frac{1}{2}2^{-n}$, therefore each one of these sequences is a Cauchy sequence. Let a_i^∞ be the limit of this sequence. By construction, we have that $\{a_i^\infty\}_{i<\omega}$ is a dense subset of $\mathfrak{M}$.

For any $i, j < \omega$, compute $d^{\mathfrak{M}}(a_i^\infty, a_j^\infty)$ as follows. Find $\ell, p < \omega$ such that $q_{\langle k_i(\ell),k_j(p)\rangle} < \frac{3}{2}$ (this implies that $a_{k_i(\ell)}$ and $a_{k_j(p)}$ are in the same copy of $\mathfrak{M}$ inside $\mathfrak{M}^*$). Now for each $o < \omega$, find the first m such that $s_{\langle k_i(\ell+o),k_j(p+o),0,m\rangle} > -\frac{1}{2}$, and then find the first n such that $s_{\langle k_i(\ell+o),k_j(p+o),n,m\rangle} \leq \frac{3}{4}$ and return $2^{-m}n$ as the oth approximation of $d^{\mathfrak{M}}(a_i^\infty, a_j^\infty)$. An easy calculation shows that $|d^{\mathfrak{M}}(a_i^\infty, a_j^\infty) - 2^{-m}n| < 2^{-m+2} \leq 2^{-o+2}$, so we have that these approximations converge uniformly in a way that only depends on o. □

Finally, we should sketch a modification of Definition 6.7 and a proof of Proposition 6.9 for arbitrary signatures that does not rely on [7]. The proof is largely the same once the definition is set up correctly. The definition of $\mathrm{HK}^*(\mathfrak{M})$ would need to contain analogs of S for each relation and function symbol, encoding their values at higher and higher precisions. There is a subtlety with regards to the modulus of uniform continuity of the predicates in question. Plugging an arbitrary relation into the definition of S will not necessarily produce a uniformly continuous function, which is required by the definition of metric structure. We have two options. We can either expand the definition of metric structure to allow non-uniformly continuous functions—which would be reasonable given the fact that the uniformity requirement is largely motivated by first-order compactness, which we are not using—or we can modify the definition of S to space out the levels of magnification more in order to keep in step with the dilation of the metric. For example, if $P(x)$ has a modulus of uniform continuity of $\sqrt{x}$, then we might modify the definition of S like this.

- $S^{\mathrm{HK}^*_P(\mathfrak{M})}((x,k),n,m) = -1$ if either $k_0 < m$, $k_1 < m$, either of x or y is not a natural number, or m is not a perfect square and
- $S^{\mathrm{HK}^*_P(\mathfrak{M})}((x,k),n,m^2) = [(2^m P^{\mathfrak{M}^*}((x,k)) - n) \vee 0] \wedge 1$ otherwise.

Such a modification would ensure that S_P is uniformly continuous, but now we need to be worried about the modulus of uniform continuity of P containing non-trivial information (in a computability theoretic sense), which surprisingly enough was not an issue up until this point.

Similar modification could be made to deal with structures in infinite signatures, although now, as in II.4.1 in [9], we would need to encode all predicates into a single predicate with an additional indexing argument, and similar issues would arise with regards to moduli of continuity.

7. Conclusion

By translating the notions of Δ_0, Σ_1, and Π_1 formulas to continuous logic (Definition 1.8), we were able to generalize the notions of Turing and enumeration reducibility to the context of $[0,1]$-valued oracles on the natural numbers, giving the notions of analog and analog enumeration reducibility (Definition 1.9). After characterizing these reduction in terms of more traditional computability theoretic notions (Theorem 1.18 and Corollary 1.20), we we demonstrated that the degree structures induced by these reductions have natural embeddings from the Turing and enumeration degrees (Proposition 2.2), and we used this relationship to give a novel characterization of the continuous degrees of [8] (Proposition 3.9). We then demonstrated that the embeddings of the Turing and enumeration degrees into the analog and analog enumeration degrees, respectively, are proper, in that they miss any sufficiently generic $[0,1]$-valued oracle (Corollary 4.4). Finally, we extended these notions to computation in an arbitrary metric structure and gave characterizations of a couple natural generalizations of r.i.c.e. relations to the context of metric structures (Theorem 6.6 and Proposition 6.9).

One avenue of future research might be to give a more precise characterization of the discrete degrees as they sit in the analog degrees (in the vein of Definition 3.7 and Question 3.8) and to clarify the relationship between the analog degrees and the collection of countable ideals of continuous degrees (Question 3.11). Another avenue might be to find more natural classes of $[0,1]$-valued oracles that fail to have discrete degree, such as possibly sufficiently random oracles (Question 4.5). Finally, one might try to extend the results of this paper to the more restrictive variants of analog and analog enumeration reduction given in Definition 5.2 (Questions 5.5 and 5.6).

Index of Definitions and Symbols

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