Special coheirs and model-theoretic trees

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Iowa State University

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UIC Logic Seminar

Combinatorial tameness in model theory

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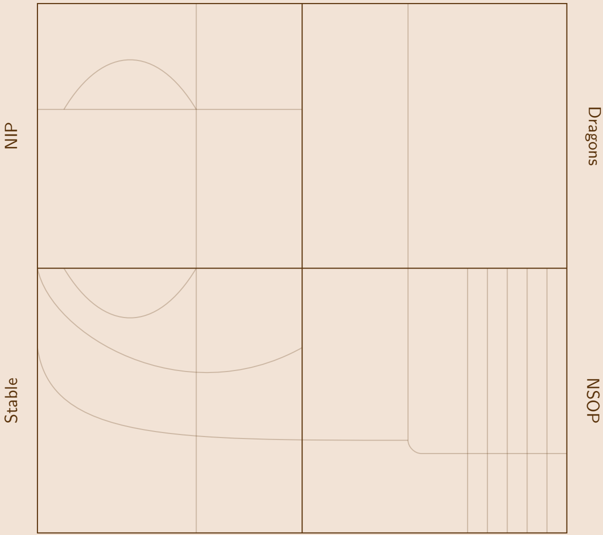
- Started with Shelah's work generalizing Morley's theorem to uncountable languages. Ballooned into a large body of work called *stability theory*. Later extended and generalized under the title of *neostability theory*.

The map: Model-theoretic adjectives



Examples:

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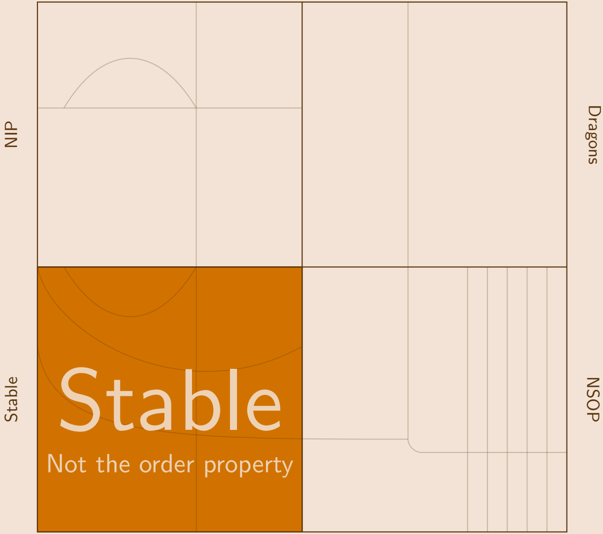

Examples:

Algebraically closed fields

Differentially closed fields

Vector spaces

Modules

Free groups

Curve graphs of surfaces

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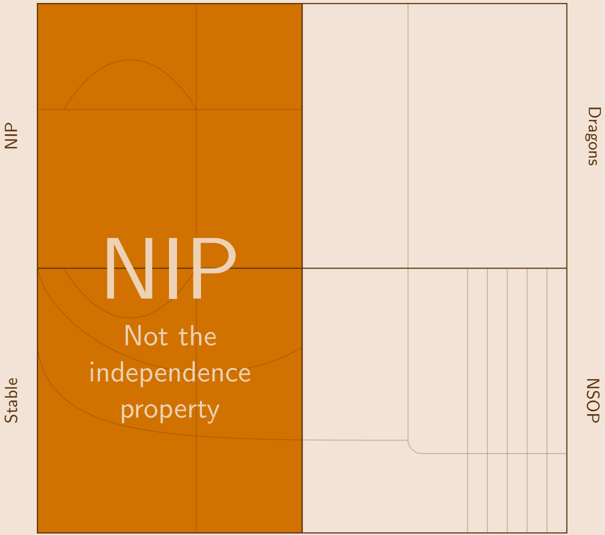

Examples:

$(\mathbb{R}, +, \cdot, <, \exp)$

$(\mathbb{Q}, +, <)$

$(\mathbb{N}, +, <)$

p-adic numbers

Alg. closed valued fields

Field of transseries

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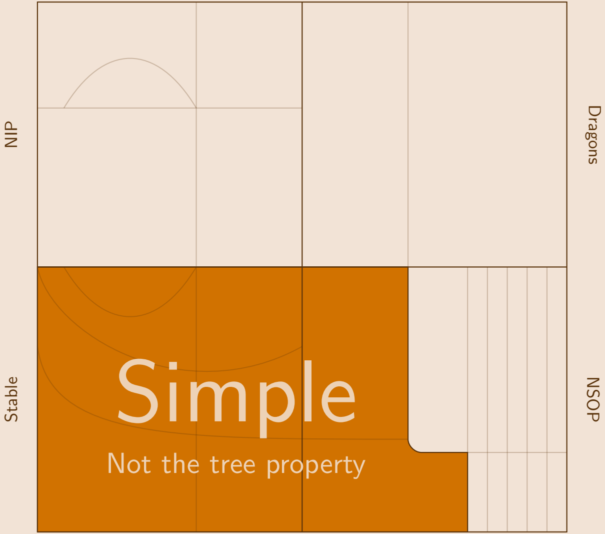

Examples:

Random graph

Pseudo-finite fields

Generic difference fields

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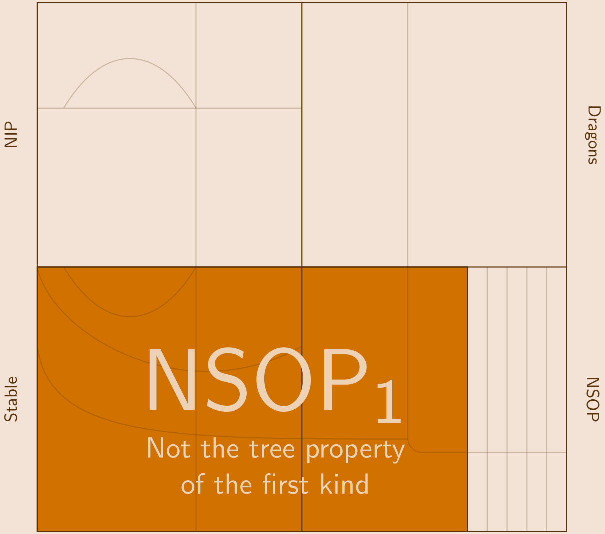

Examples:

Generic vector spaces with bilinear forms

Generic binary function

Generic parameterized equivalence relation

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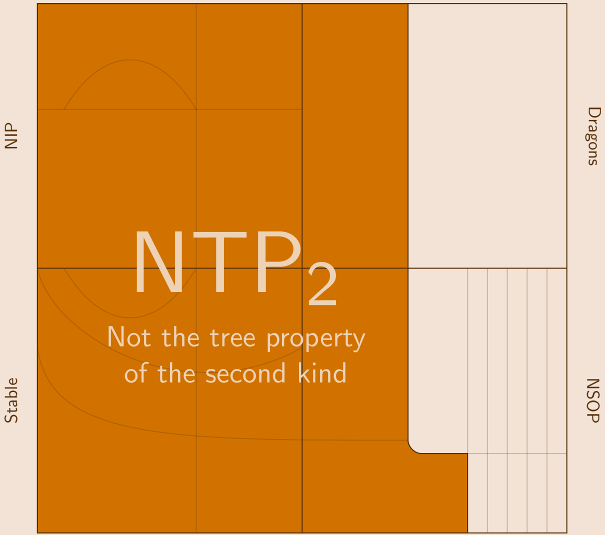

Examples:

Ultraproduct of $\mathbb{Q}_p$

Densely ordered random graph

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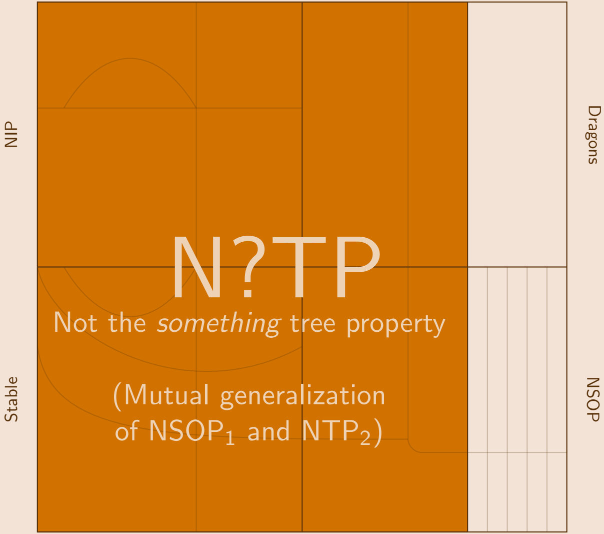

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Generic vector space with bilinear form over NIP or NTP_2 field ($\mathbb{R}$, $\mathbb{Q}_p$, etc.)

Generic linear order
$+$
binary function

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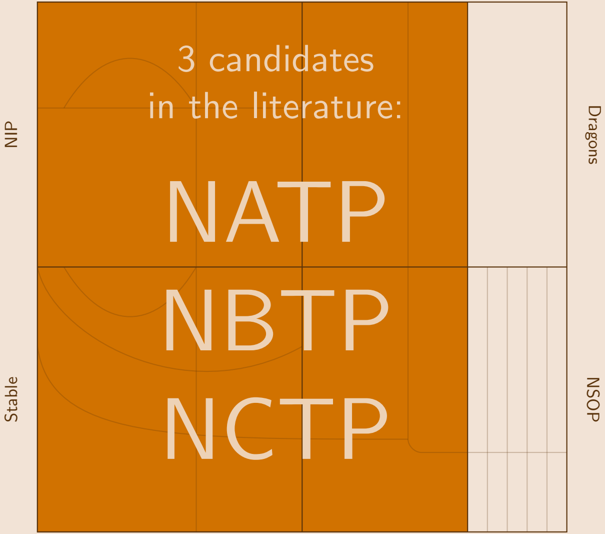

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Simplicity: The tree property in model theory

A formula $\varphi(x, y)$ has the *k-tree property* if there is a tree $(c_\sigma)_{\sigma \in \omega^{<\omega}}$ of parameters such that

- paths are consistent: $\{\varphi(x, c_{\alpha \upharpoonright n}) : n < \omega\}$ for $\alpha \in \omega^\omega$,
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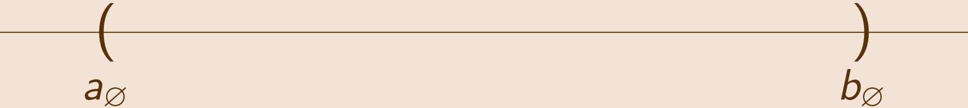

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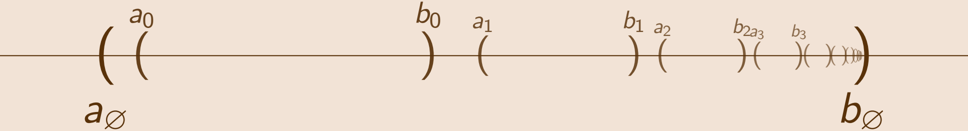

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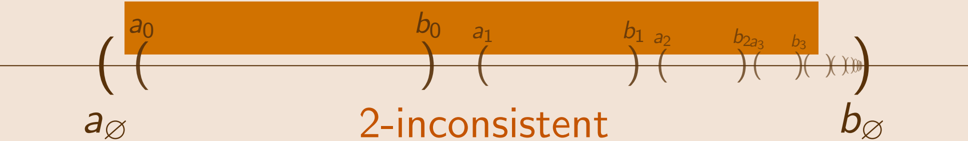

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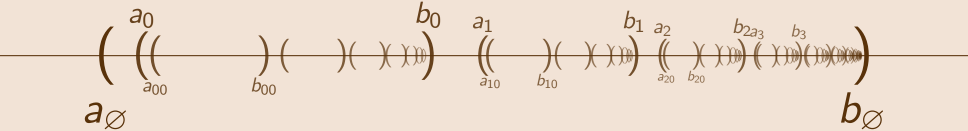
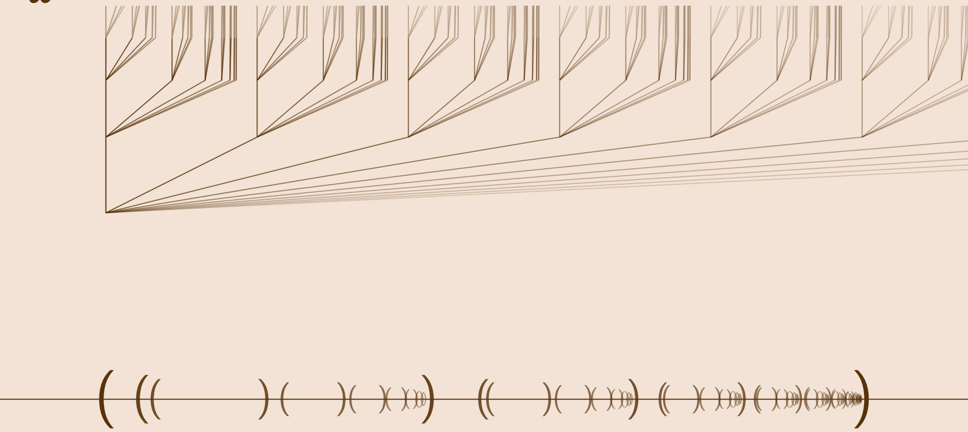


The tree in the tree property

$\omega^{<\omega}$



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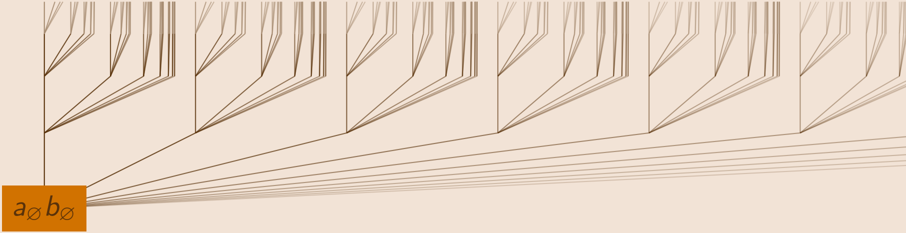

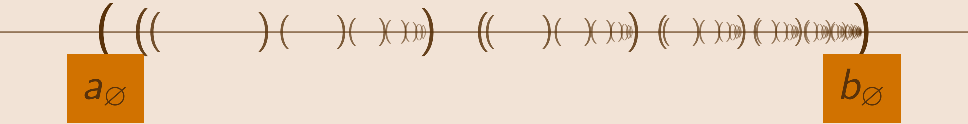

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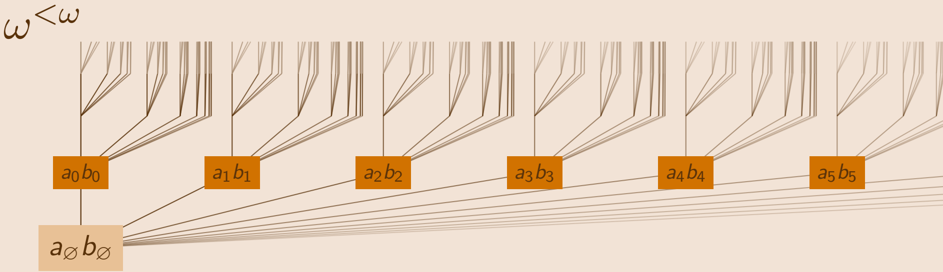

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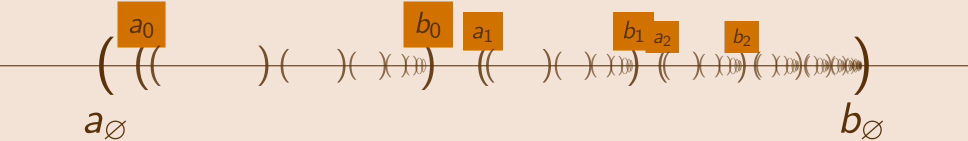

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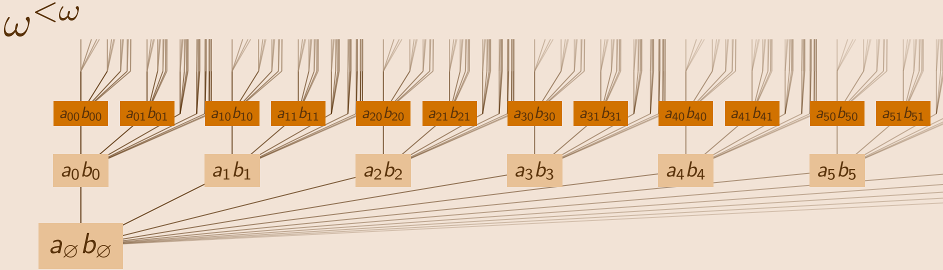

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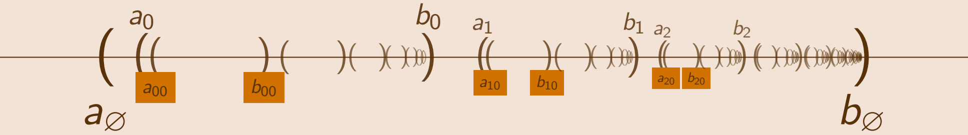

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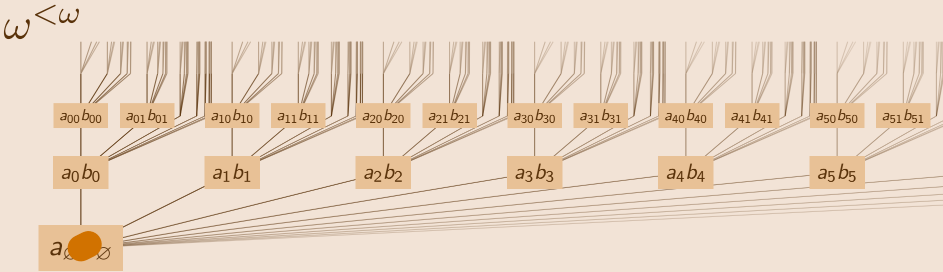

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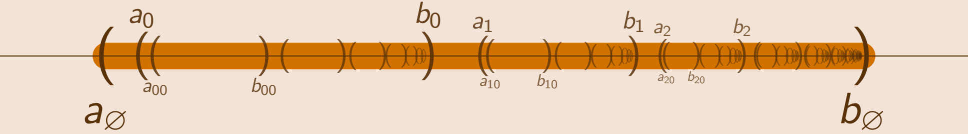

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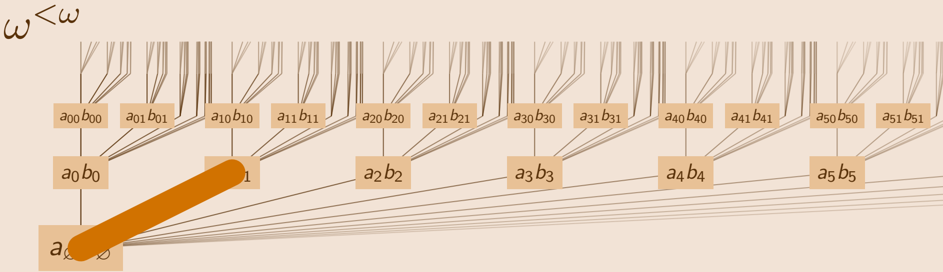

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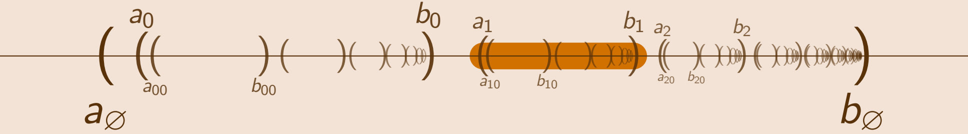

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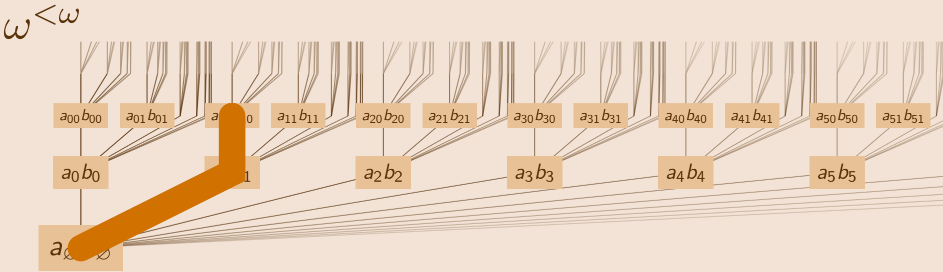

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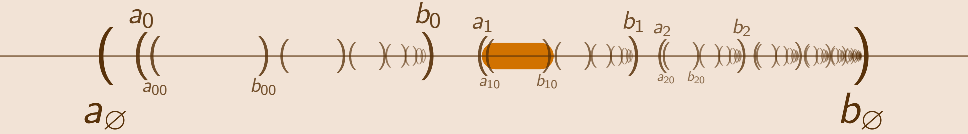

SOP_1: The tree property of the first kind

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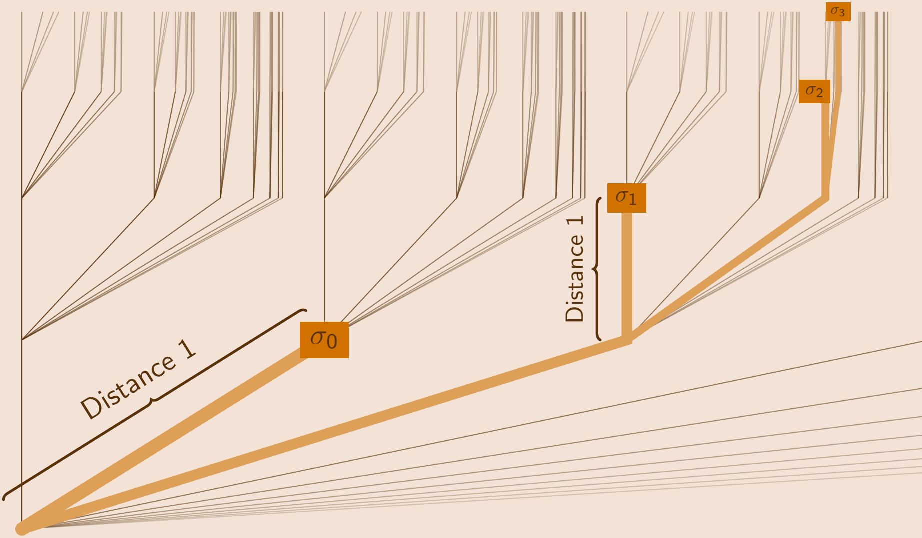
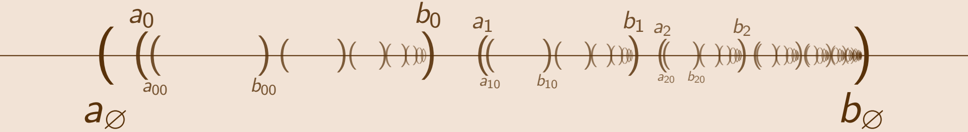
A short-toothed right-comb



$(\mathbb{Q}, <)$ has 2-SOP$_1$

In our tree in $(\mathbb{Q}, <)$, any pair of incomparable elements are inconsistent.



Hence any short-toothed right-comb is 2-inconsistent.

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- $a_0, a_1, \ldots$ is the *Morley sequence generated by* $\mathcal{U}$.

SOP_1 in terms of coheirs

Given a coheir $\mathcal{U}$ over a model M, a formula $\varphi(x, y)$ *k-divides along* $\mathcal{U}$ if whenever $b_0, b_1, \ldots$ is a Morley sequence generated by $\mathcal{U}$, $\{\varphi(x, b_i) : i < \omega\}$ is k-inconsistent.

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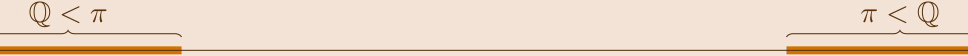
Theorem (Kaplan, Ramsey)

T has SOP_1 if and only if there is a model M, two coheirs $\mathcal{U}$ and $\mathcal{V}$ (extending the same type), and a formula $\varphi(x, y)$ such that $\varphi(x, y)$ divides along $\mathcal{U}$ but not along $\mathcal{V}$.

Coheir witnesses of SOP_1 in $(\mathbb{Q}, <)$

Two non-trivial coheirs of the 2-type living in the cut at π over $\mathbb{Q}$:



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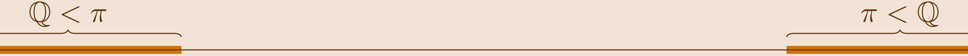

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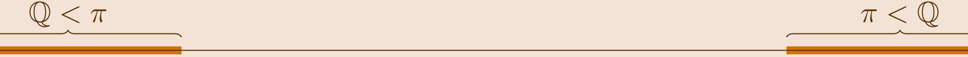

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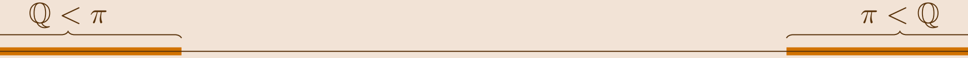
The formula $(a < x < b)$ divides along $\mathcal{U}_{\mathsf{below}}$ but not along $\mathcal{U}_{\mathsf{pinch}}$.



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Morley sequence generated by $\mathcal{U}_{\mathrm{pinch}}$
Consistent

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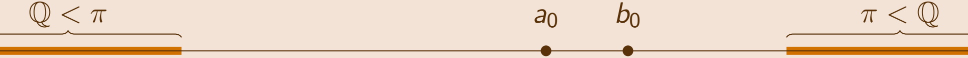
$\mathcal{U}_{\text{below}}$ has a special property. The Morley sequence it generates



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Heir-coheirs

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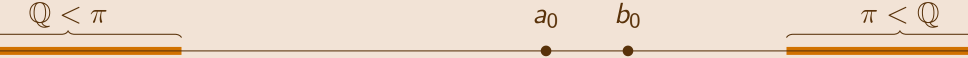

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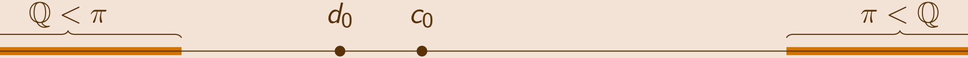

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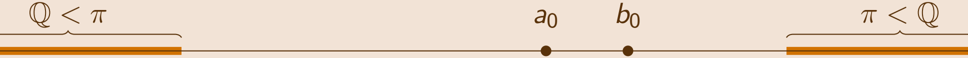

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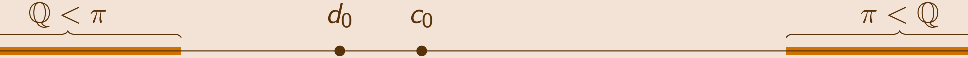

Generated by $\mathcal{U}_{\text{above}}$

This is non-trivial. $\mathcal{U}_{\text{pinch}}$ *does not* have this property.

TP_2 in terms of heir-coheirs

Definition

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Theorem (Chernikov, Kaplan)

T has TP_2 if and only if there is a model M, a formula $\varphi(x, b)$, and an M-heir-coheir $\mathcal{U}$ extending the type of b over M such that $\varphi(x, b)$ divides over M but does not divide along $\mathcal{U}$.

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$\mathcal{U}$ is an *M-heir-coheir* if whenever b realizes $\mathcal{U}$ over $M \cup A$, there is an M-coheir $\mathcal{V}$ such that A realizes $\mathcal{V}$ over $M \cup b$.

A formula $\varphi(x, b)$ *k-divides over M* if there is a sequence $(b_i)_{i<\omega}$ of realizations of the type of b over M such that $\{\varphi(x, b_i) : i < \omega\}$ is k-inconsistent.

Theorem (Chernikov, Kaplan)

T has TP_2 if and only if there is a model M, a formula $\varphi(x, b)$, and an M-heir-coheir $\mathcal{U}$ extending the type of b over M such that $\varphi(x, b)$ divides over M but does not divide along $\mathcal{U}$.

DLO (theory of $(\mathbb{Q}, <)$) is NTP_2.

N?TP via a new Kim's lemma?

Dividing lines tend to have three characterizations: Combinatorial, some kind of local character, and a version of Kim's lemma.

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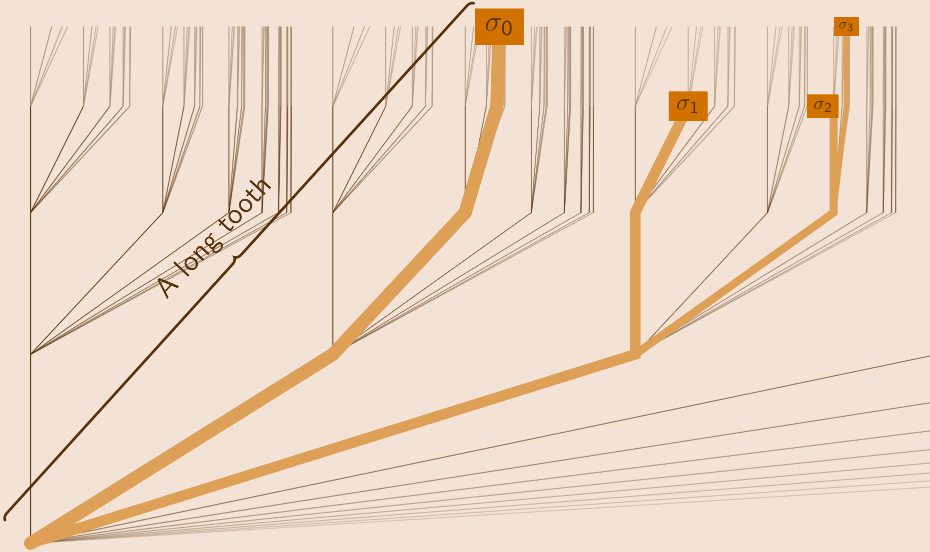
Mutchnik established the following in his proof that $\mathrm{NSOP}_1 = \mathrm{NSOP}_2$.

Theorem (Mutchnik)

The above condition without the switcheroo is equivalent to SOP_1.

A right-comb



Characterization of CTP

Theorem (H.)

A theory has k-CTP if and only if there is a model M, a formula $\varphi(x, b)$, and an M-heir-coheir $\mathcal{U}$ and an M-coheir $\mathcal{V}$ extending the type of b over M such that $\varphi(x, b)$ k-divides along $\mathcal{V}$ but does not divide along $\mathcal{U}$.

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Does k-CTP imply 2-CTP?

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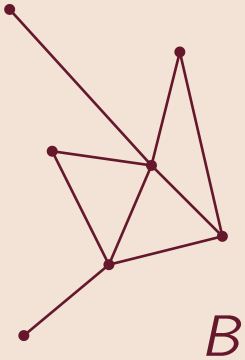
We also have the following alphabetically frustrating implication:

$$\text{ATP} \Rightarrow \text{CTP} \Rightarrow \text{BTP}$$

where the *antichain tree property* or *ATP* is another candidate for ?TP, introduced by Ahn and Kim.

What's special about heir-coheirs?

If $\mathcal{U}$ is an M-heir-coheir and B is some configuration of realizations of $\mathcal{U}$ over M, then we can find a clone B' of B with the property that every element of B' realizes $\mathcal{U}$ over $M \cup B$.



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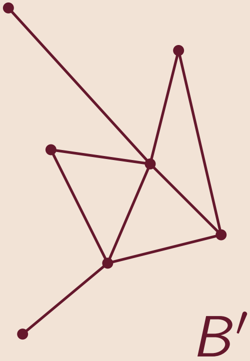

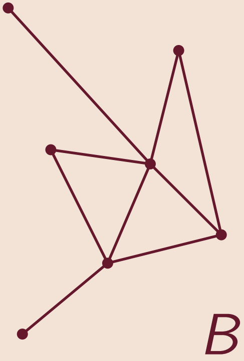

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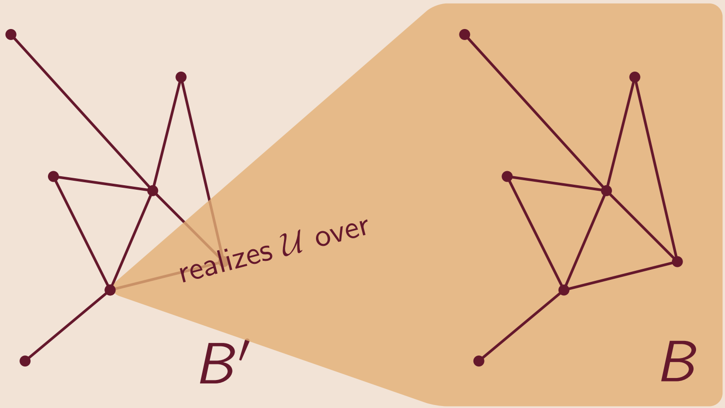

CTP from heir-coheir $\mathcal{U}$ and coheir $\mathcal{V}$

Realize $\mathcal{U}$ •

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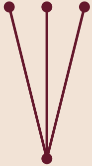

Realize $\mathcal{V}$

CTP from heir-coheir $\mathcal{U}$ and coheir $\mathcal{V}$

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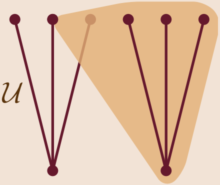

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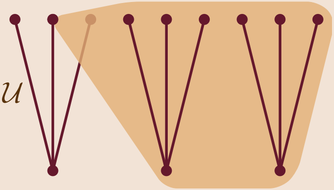

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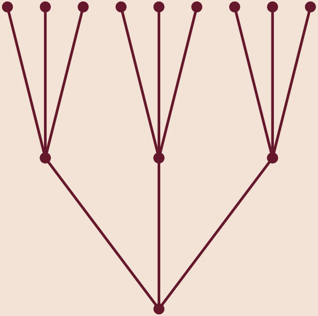

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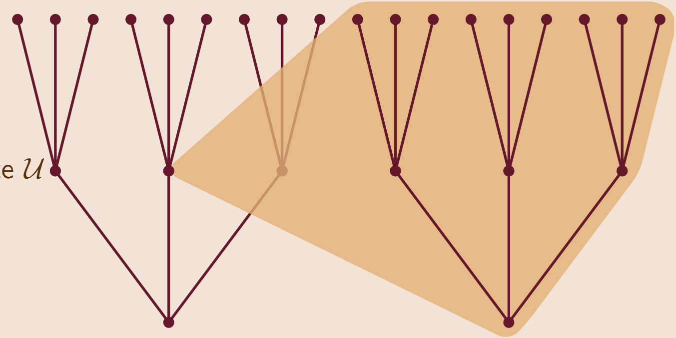

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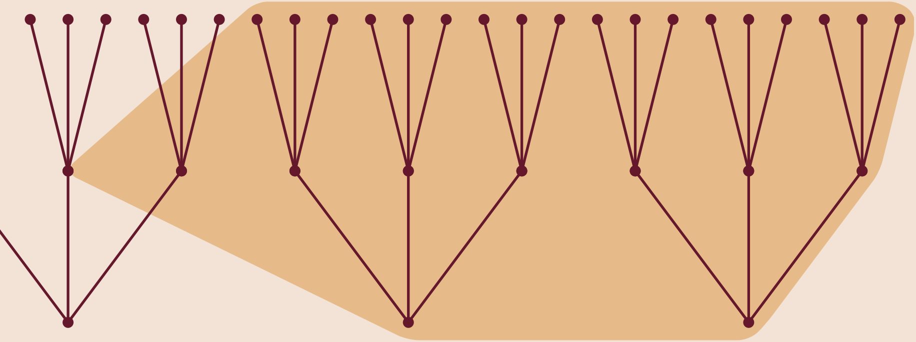

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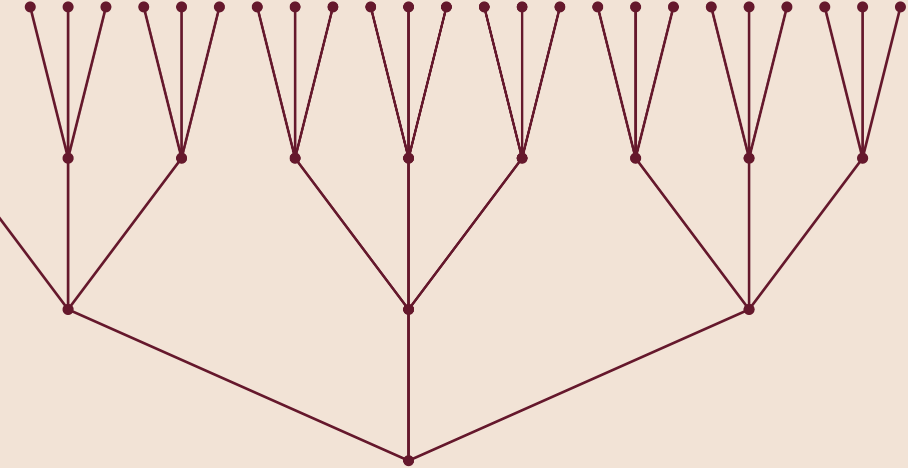

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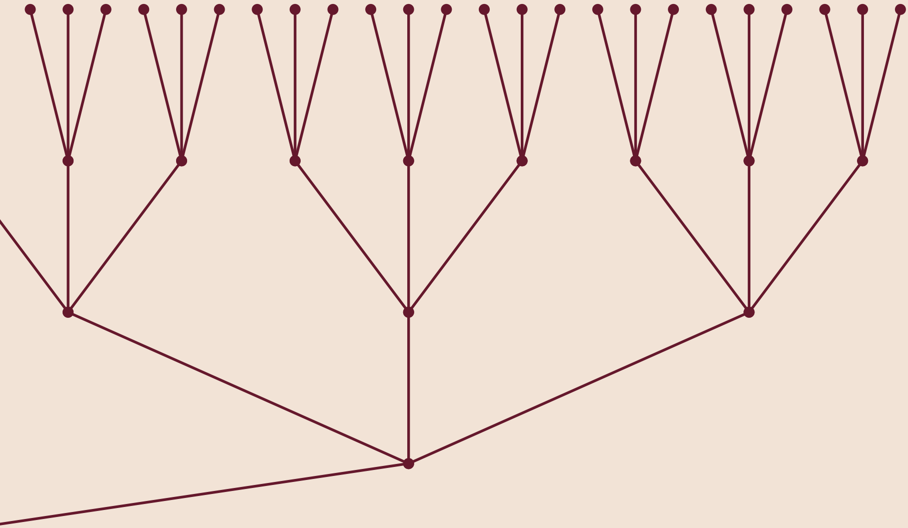

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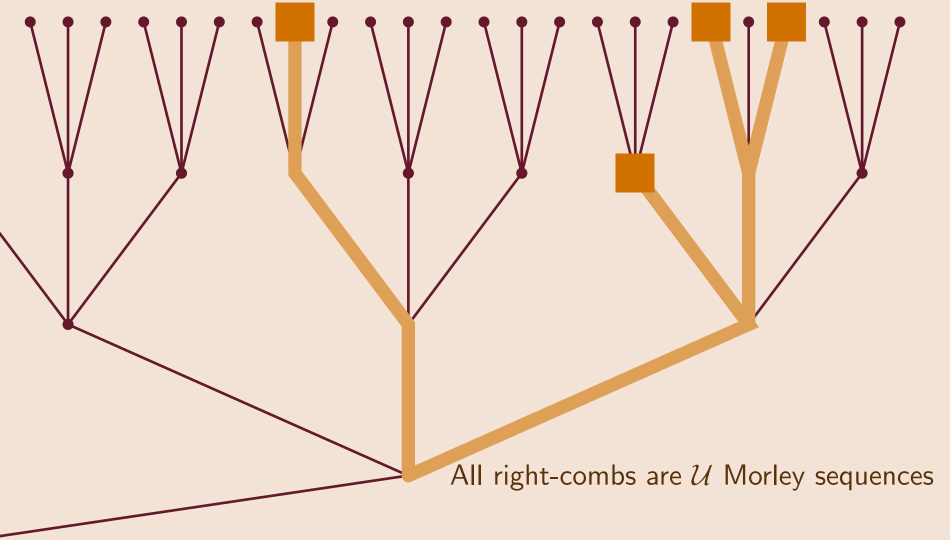

All right-combs are $\mathcal{U}$ Morley sequences

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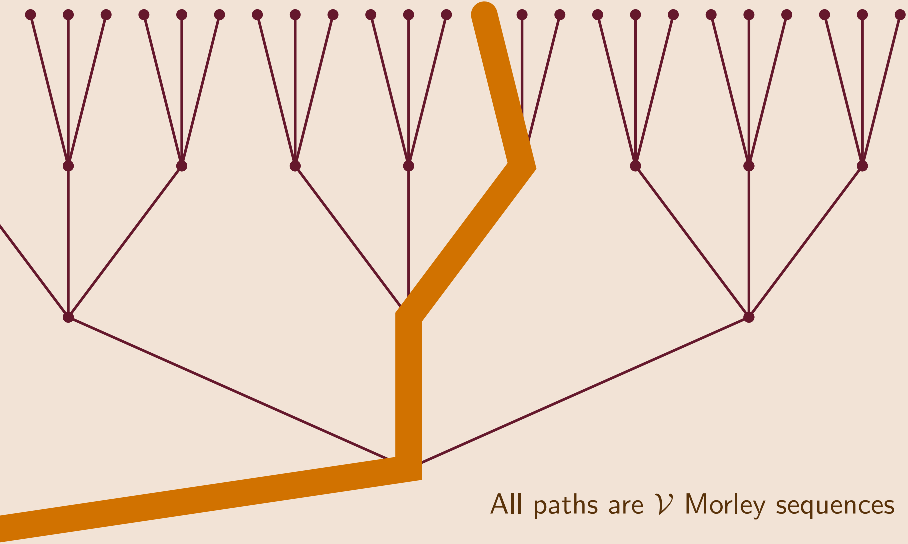

All paths are $\mathcal{V}$ Morley sequences

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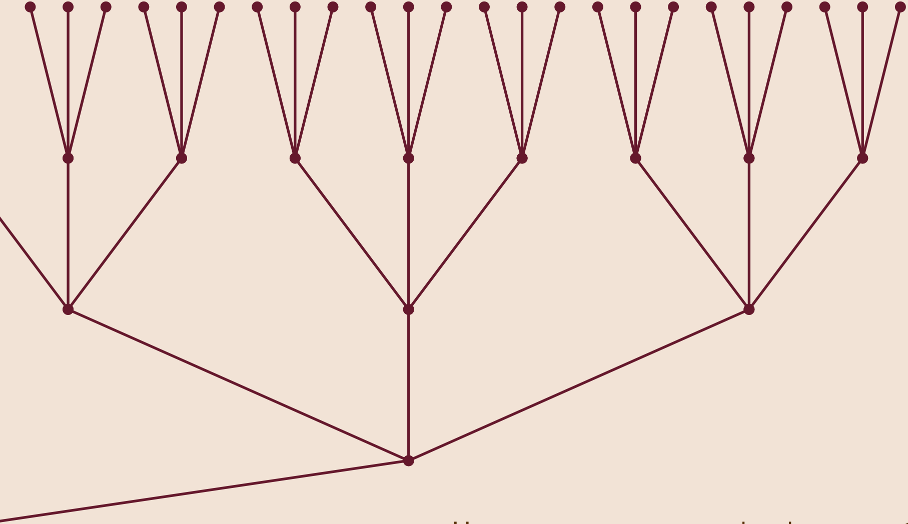

Use compactness to make the tree $\omega^{<\omega}$

Forcing

Whence do heir-coheirs come?

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If $\mathcal{U}$ is a coheir over M and $N \succ M$ is a sufficiently saturated elementary extension, then $\mathcal{U}$ is an heir-coheir over N.

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There are many heir-coheirs over $(\mathbb{Q}, <)$ (any non-realized cut). Is this generalizable?

Miniaturizing the saturation argument

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Argue that if $\mathcal{U}$ extends the type we built and a realizes $\mathcal{U}$ over Mb, then every formula in the type of b over Ma is already finitely satisfiable in M by construction. □

Thank you

Comb definitions

Short-toothed right-combs are defined inductively:

- $\varnothing$ is a short-toothed right-comb.
- X is a short-toothed right-comb, every element of X extends $\sigma \frown j$, and $i < j$, then $X \cup \{\sigma \frown i\}$ is a short-toothed right-comb.

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The miniaturized argument as a blueprint for CTP

That proof is a forcing argument: We have a set of conditions that we need to satisfy and we are free to satisfy them generically.

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The comb tree property (even on $2^{<\omega}$ rather than $\omega^{<\omega}$) gives you precisely what you need to generically build an heir-coheir $\mathcal{U}$ that is 'shadowed' by a coheir $\mathcal{V}$ such that the given formula divides along $\mathcal{V}$ but not along $\mathcal{U}$.

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(Draw on chalkboard.)

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Forcing with comb trees II

The second bullet point now ensures that

$$\mathcal{F} \cup \left\{ \bigcup_{i<\omega} (\text{cone above } \sigma_i \frown 0) \right\}$$

generates a non-trivial filter, which can be extended to an ultrafilter $\mathcal{U}$ whose elements are all somewhere dense.

The third bullet point ensures that $\mathcal{U}$ is in fact an heir-coheir and the extra set added to $\mathcal{F}$ ensures that $\varphi(x, y)$ does not divide along $\mathcal{U}$.

Finally, let $\mathcal{V}$ be any non-principal ultrafilter on $\{b_{\sigma_i} : i < \omega\}$. By construction, $\varphi(x, y)$ will divide along $\mathcal{V}$. Furthermore, the third bullet point will ensure that $\mathcal{U}$ and $\mathcal{V}$ extend the same type over M, so we have the required failure of Kim's lemma for coheirs and heir-coheirs.

Forcing with comb trees III