Measurability properties of model-theoretic invariant types

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Invariant types

Model-theoretic types

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- In 'good' M and for small B, it will be the case that $\bar{a} \equiv_B \bar{a}'$ if and only if there is an automorphism σ of M fixing B pointwise such that $\sigma \cdot \bar{a} = \bar{a}'$.

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- Again: $\bar{a} \equiv_B \bar{a}'$ iff there is a $\sigma \in \mathrm{Aut}(U/B)$ such that $\sigma \cdot \bar{a} = \bar{a}'$.

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- A global type $p(\bar{x})$ is *B-invariant* if $\sigma \cdot p(\bar{x}) = p(\bar{x})$ for every $\sigma \in \mathrm{Aut}(U/B)$.
- Equivalent to saying that for any $\bar{c} \equiv_B \bar{c}'$, $\varphi(\bar{x}, \bar{c}) \in p(\bar{x})$ iff $\varphi(\bar{x}, \bar{c}') \in p(\bar{x})$.

Examples: Definable types

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- For any small model $M \preceq U$ and type $q(\bar{x}) \in S_{\bar{x}}(M)$, there is at most one M-definable $p(\bar{x}) \supseteq q(\bar{x})$.
- Important in stability theory. A theory is stable if and only if every type over a model has a definable global extension.

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- DLO has $\leq 2^{|M|}$ M-invariant types (cuts) and RG has $2^{2^{|M|}}$ M-invariant types (ultrafilters).
- A theory is NIP iff it has $\leq 2^{|M|}$ M-invariant types over every M (Poizat and Shelah).

Measurability

- Given a B-invariant type $p(\bar{x})$ and a B-formula $\varphi(\bar{x}, \bar{y})$, let

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- Borel-definability is nice for reasons I'm not going to get into now.

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- When does every type over a model extend to a Borel-definable type?
- When are Borel-definable types dense?
- Does every simple theory admit a non-trivial Borel-definable type?

Borel propositional theories

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- Let $(A_r)_{r\in\mathbb{R}}$ be a family of atomic propositions. Let F be the set of propositional formulas built out of the A_r's. F has a natural Polish space structure.

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Given any first-order structure M and type $p(\bar{x}) \in S_{\bar{x}}(M)$, there are propositional theories (in the above sense) that correspond to the problem of finding M-invariant global extensions of $p(\bar{x})$ and to finding M-coheirs extending $p(\bar{x})$.

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Question: When does a Borel set of axioms have a Borel completion?

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Question: When does a Borel set of axioms have a Borel completion?

- It is easy to show that for any Borel $T_0 \subseteq F$, the smallest propositional theory $T \supseteq T_0$ is Σ^1_1. In particular, any complete theory with a Borel axiomatization is Borel.
- The set of codes of Σ^1_1 sets $T_0 \subseteq F$ such that T_0 has a Borel completion is Σ^1_3. Is it Σ^1_3-complete?

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Proof.

Let $C \subseteq 2^F$ be the set of consistent sets of formulas. C is Π^1_1 on Σ^1_1 (i.e. for any Polish space X and any Σ^1_1 set $A \subseteq X \times F$, the set $\{x \in X : A_x \in C\}$ is Π^1_1).

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Let $C \subseteq 2^F$ be the set of consistent sets of formulas. C is Π^1_1 on Σ^1_1 (i.e. for any Polish space X and any Σ^1_1 set $A \subseteq X \times F$, the set $\{x \in X : A_x \in C\}$ is Π^1_1). By the first reflection theorem, we can find a Borel set $T_1 \supseteq T_0$. The closure of T_1 under logical consequence is also Σ^1_1, and so we can by induction find an ascending chain $T_0 \subseteq T_1 \subseteq T_2 \subseteq \dots$ of Borel sets in C. $T' = \bigcup_{n<\omega} T_n$ is the required set. ☐

This can be applied in the model-theoretic context to 'partial invariant types' (the analog of incomplete propositional theories),

One nice thing

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This can be applied in the model-theoretic context to 'partial invariant types' (the analog of incomplete propositional theories), but this also doesn't seem to be a path to actually building complete Borel-definable types.

More incremental progress

A hopefully easier question

In applications, what often matters is that the sets $D_p^\varphi \subseteq S_{\bar{y}}(B)$ are measurable with regards to some regular Borel probability measure μ on $S_{\bar{y}}(B)$. (And often we really only need this for one particular formula φ.)

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In principle, all we need to do is the following:

- Fix a measure μ on $S_{\bar{y}}(B)$.
- Fix a type $p(\bar{x}) \in S_{\bar{x}}(B)$ of which we would like a B-invariant extension satisfying that D_p^φ is μ-measurable.

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- Then by compactness, the desired B-invariant global extension will exist (because the set of B-invariant types in $S_{\bar{x}}(U)$ is closed).

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- I'm able to show something similar about invariant types in first-order theories, but since it works for ultrafilters this might not say that much about the conundrum.
- On the other hand, the argument seems to use special properties of the invariant type setting, and *might* not work for arbitrary Borel sets of propositional axioms.

A partial answer II

Proposition

Fix a set of parameters B and a type $p(\bar{x}) \in S_{\bar{x}}(B)$. Assume that $p(\bar{x})$ has a global B-invariant extension. Then for any B-formula $\varphi(\bar{x}, \bar{y})$ and any Polish topology τ on $S_{\bar{y}}(B)$ refining the standard topology, $p(\bar{x})$ has a global B-invariant extension $q(\bar{x})$ such that for any open $U \in \tau$, $U \cap D_p^{\varphi}$ has the perfect set property.

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The same hold with 'B-coheir' in place of 'B-invariant.'

Can also do this for all formulas at the same time, but messy.

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Lemma 1

There is a sequence $(A_n)_{n<\omega}$ of sets of pairs $(\psi^+(\bar{y}), \psi^-(\bar{y}))$ of M-formulas such that (F^+, F^-) is admissible if and only if for every n, there is a $(\psi^+, \psi^-) \in A_n$ such that $\psi^+(\bar{y}) \in r(\bar{y})$ for every $r \in F^+$ and $\psi^-(\bar{y}) \in r(\bar{y})$ for every $r \in F^-$.

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  - for any $\bar{c}_0, \dots, \bar{c}_{n-1}$ satisfying $\psi^+(\bar{y})$ and $\bar{d}_0, \dots, \bar{d}_{n-1}$ satisfying $\psi^-(\bar{y})$, there is a $j < k$ with $\bigwedge_{i<n} \varphi(\bar{a}_j, \bar{c}_i) \wedge \bigwedge_{i<n} \neg\varphi(\bar{a}_j, \bar{d}_i)$.

A second lemma

Lemma 2

Fix a type $p(\bar{x}) \in S_{\bar{x}}(M)$, an M-formula $\varphi(\bar{x}, \bar{y})$, and an admissible pair (F^+, F^-) of closed subsets of $S_{\bar{y}}(M)$. For any uncountable closed set $H \subseteq S_{\bar{y}}(M)$, there is an admissible pair (G^+, G^-) of closed subsets of $S_{\bar{y}}(M)$ such that $G^+ \supseteq F^+$, $G^- \supseteq F^-$, and $H \cap (G^+ \cup G^-)$ is uncountable.

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Proof.

(Board.) ☐

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- By compactness, H is closed in the ordinary topology on $S_{\bar{y}}(M)$. Apply Lemma 2 to F_n^+, F_n^-, and H to get (F_{n+1}^+, F_{n+1}^-).

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- By compactness, H is closed in the ordinary topology on $S_{\bar{y}}(M)$. Apply Lemma 2 to F_n^+, F_n^-, and H to get (F_{n+1}^+, F_{n+1}^-).
- Finally, apply compactness to build an M-coheir $q(\bar{x}) \supseteq p(\bar{x})$ satisfying that for any $\bar{c}$ with $\mathrm{tp}(\bar{c}/M) \in \bigcup_{n<\omega} F_n^+$, $\varphi(\bar{x}, \bar{c}) \in q(\bar{x})$ and for any $\bar{c}$ with $\mathrm{tp}(\bar{c}/M) \in \bigcup_{n<\omega} F_n^-$, $\neg\varphi(\bar{x}, \bar{c}) \in q(\bar{x})$.

Final thoughts

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- It might be possible in the case of specific theories to analyze the possible measures of admissible pairs (F^+, F^-) relative to a given fixed measure μ.
- Getting a general characterization seems unlikely to me thought, except maybe in the context of model-theoretically tame theories (simple, NTP_2, etc.).

Thank you