Generic Stability and Randomizations

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Joint work with Gabriel Conant and Kyle Gannon.

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- Example of an *invariant type*: A coherent way of building a type over any elementary extension. Examples: Definable types over models of PA, types induced by ultrafilters on structures (*coheirs*), etc.

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- Example of an *invariant type*: A coherent way of building a type over any elementary extension. Examples: Definable types over models of PA, types induced by ultrafilters on structures (*coheirs*), etc.
- $\mathrm{Av}(a_{<\omega})$ has a particularly nice property: If you realize it iteratively to get $(b_i)_{i<\omega}$, then $b_{<\omega}$ also has the finite-cofinite property and has $\mathrm{Av}(b_{<\omega}) = \mathrm{Av}(a_{<\omega})$. These types are called *generically stable*.

Aside: Interesting Open Question

- Given M-invariant types $p(x)$ and $q(y)$, there is a unique M-invariant type $p \otimes q(x, y)$ satisfying that if $ab \models p \otimes q{\restriction}N$ (for some elementary extension $N \succeq M$), then $a \models p{\restriction}Nb$ and $b \models q{\restriction}N$.

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- Surprisingly obnoxious to resolve in general.

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- Invariant measures are coherent procedures for 'randomly generating a type' over any larger model.

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Theorem (Conant, Gannon, H.)

The Morley product of measures can fail to be associative in a strong way: There are invariant measures μ, ν, and λ such that $\mu \otimes \nu$, $\nu \otimes \lambda$, $(\mu \otimes \nu) \otimes \lambda$, and $\mu \otimes (\nu \otimes \lambda)$ all exist and look somewhat tame yet $(\mu \otimes \nu) \otimes \lambda \neq \mu \otimes (\nu \otimes \lambda)$.

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$\sum_{r \in 2^\omega} 0 = 0 \neq 1$. □

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- (Conant, Gannon) This fails in insufficiently nice theories. Also, in many theories, such as PA and ZFC, there aren't any definable coheirs.

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A measure μ is a *frequency interpretation measure* (or *fim* measure) if whenever "$(a_i)_{i<\omega}$ is a sequence generated by iteratively realizing μ," the quantity $\frac{1}{n}|\{i < n : \varphi(a_i, b)\}|$ limits to $\mu(\varphi(x, b))$ with probability 1.

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- Not too hard to show that any fim measure is a definable coheir.
- (Conant, Gannon) An invariant type $p(x)$ is generically stable if and only if $\delta_p(x)$ is fim.
- But is this really the right notion of generic stability for measures?

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- Continuous logic is in a strong sense equivalent to earlier approaches to real-valued logics, such as $[0, 1]$-valued Łukasiewicz logic, but the associated model-theoretic approach is largely new.
- Everything* familiar from ordinary model theory generalizes to continuous model theory: Compactness, Löwenheim–Skolem, Craig interpolation, Ryll-Nardzewski, Lindström, stability theory, etc.

*Some exceptions may apply.

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- In particular, if $\mu(x)$ is a definable invariant measure, then for any formula $\varphi(x, y)$, we can think of F_μ^φ as being a formula in the sense of continuous logic.
- This allows us to quantify over expressions involving F_μ^φ.

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- (Conant, Gannon, H.) There is a definable coheir p with the property that r_{δ_p} is not a coheir.
- fim and generic stability?

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$$\chi^{\varphi}_{\nu,n}(x_1 \dots x_n) = \sup_y \left| \frac{1}{n}(\underbrace{\varphi(x_1,y) + \cdots + \varphi(x_n,y)}_{\substack{\text{True is 1.}\\ \text{False is 0.}}}) - F^{\varphi}_{\nu}(\mathrm{tp}(y/A)) \right|.$$

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- $\chi^{\varphi}_{\nu,n}(\bar{x})$ is a formula in the sense of continuous logic (because $\nu(x)$ is definable).

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- $\mu(x)$ is fim iff for every $\varphi(x,y)$, $\lim_{n\to\infty} \int \chi^{\varphi}_{\mu,n}(\bar{x}) d\mu^{\otimes n}(\bar{x}) = 0$.

fim and Generic Stability in the Randomization II

Fix an A-definable measure $\mu(x)$ in T. Let $r_\mu(x)$ be the corresponding type in $(T_A)^R$.

- $\mu(x)$ is fim iff for every $\varphi(x,y)$, $\lim_{n\to\infty} \int \chi^{\varphi}_{\mu,n}(\bar{x}) d\mu^{\otimes n}(\bar{x}) = 0$.
- $r_\mu(x)$ is generically stable iff for every $\varphi(x,y)$ (from T), $\lim_{n\to\infty} r_\mu^{\otimes n}\left(\chi^{E[\varphi]}_{r_\mu,n}(\bar{x})\right) = 0$. (Uses QE down to $E[\varphi]$.)

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$$r_\mu^{\otimes n}\left(\chi^{E[\varphi]}_{r_\mu,n}(\bar{x})\right) = r_\mu^{\otimes n}\left(\sup_y \left| E\left[\frac{\varphi(x_1 y) + \cdots + \varphi(x_n y)}{n} - F^{\varphi}_{\mu}(y)\right]\right|\right)$$

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First $\leq$ second by Jensen's inequality, so if μ is fim, then r_μ is generically stable, but will it reverse?

Thank you