Forcing with model-theoretic trees

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University of Maryland

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University of Maryland Logic Seminar

The tree property in model theory

A formula $\varphi(x, y)$ has the *k-tree property* if there is a tree $(c_\sigma)_{\sigma \in \omega^{<\omega}}$ of parameters such that

- paths are consistent: $\{\varphi(x, c_{\alpha \upharpoonright n}) : n < \omega\}$ for $\alpha \in \omega^\omega$,
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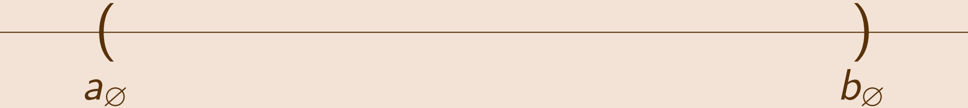

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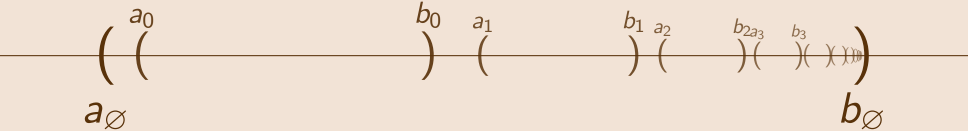

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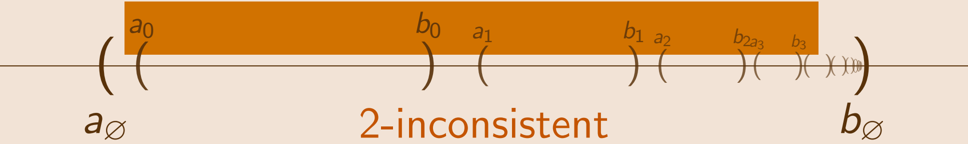

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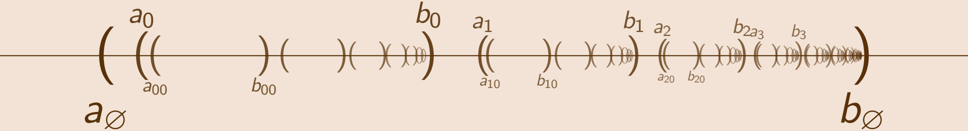

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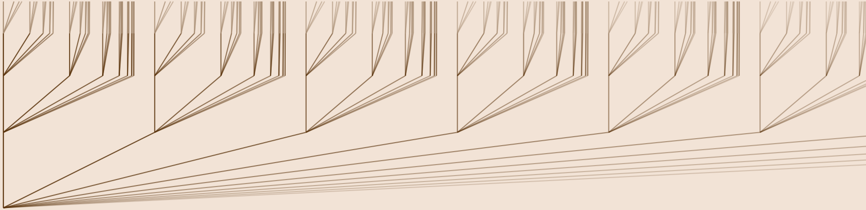
$\omega^{<\omega}$



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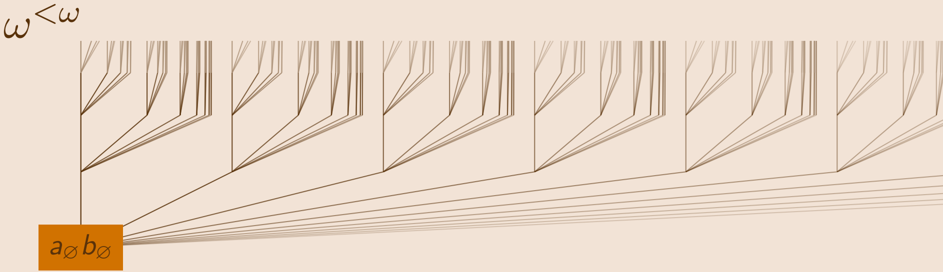

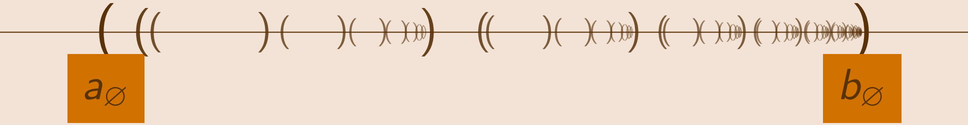

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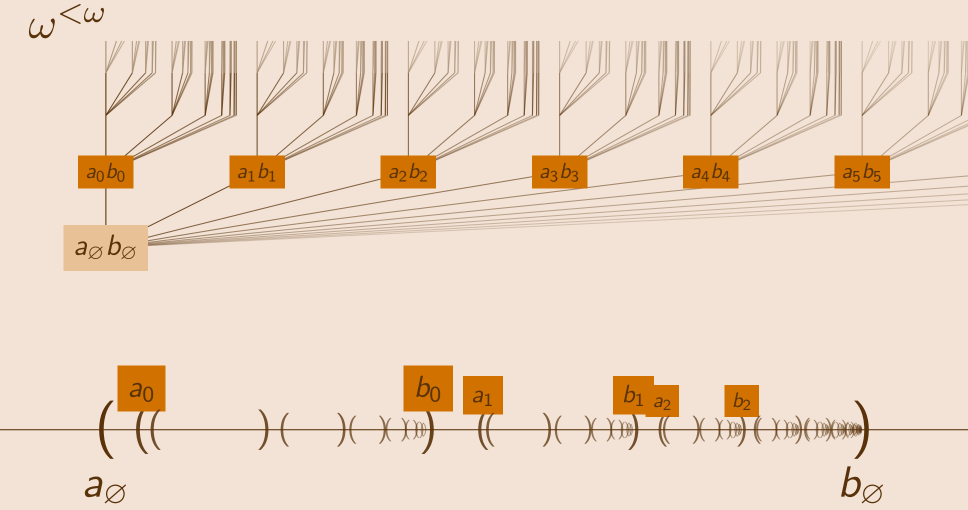

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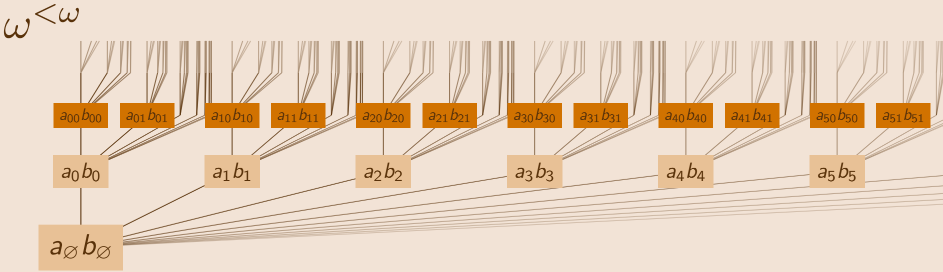

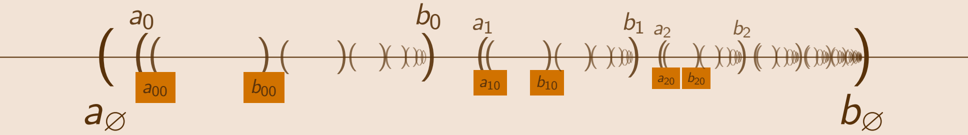

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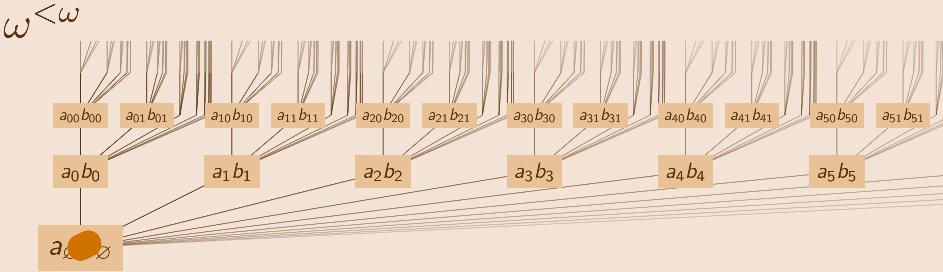

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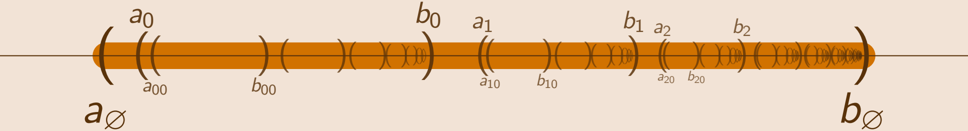

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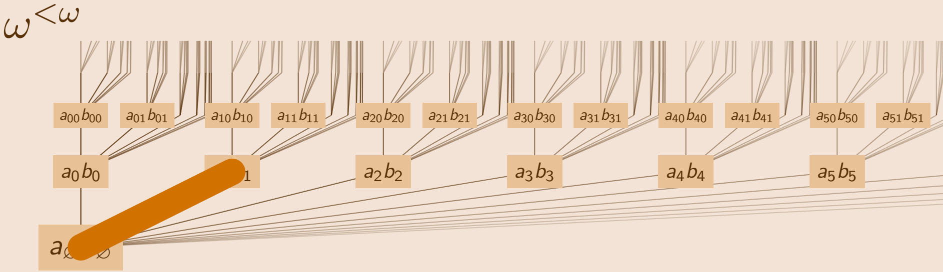

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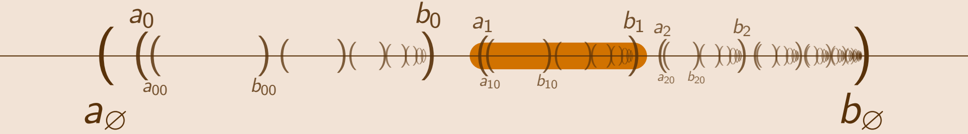

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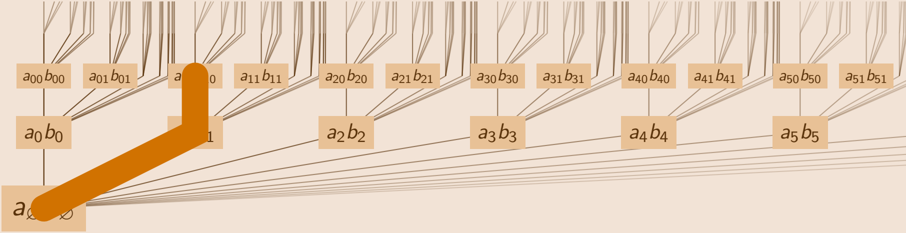

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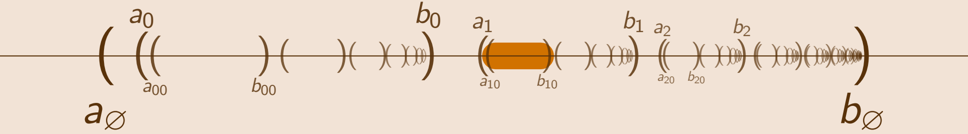

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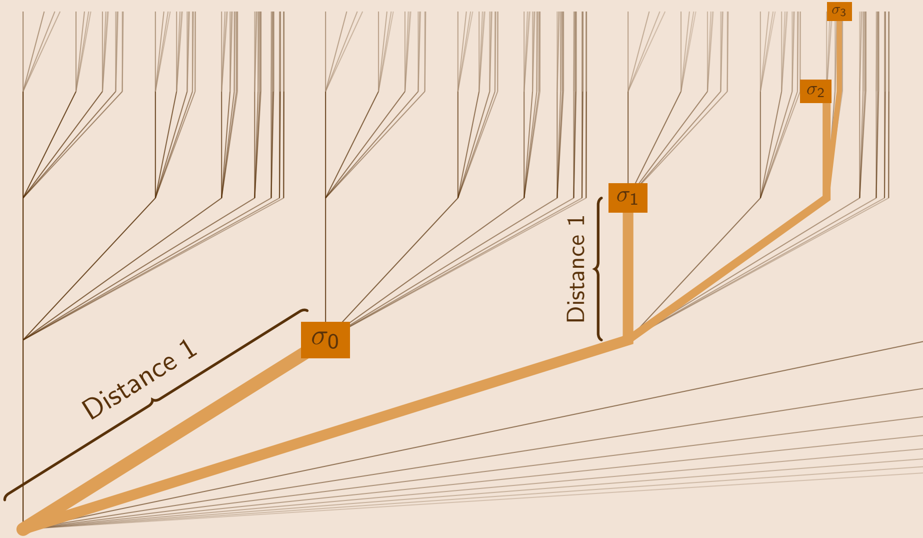
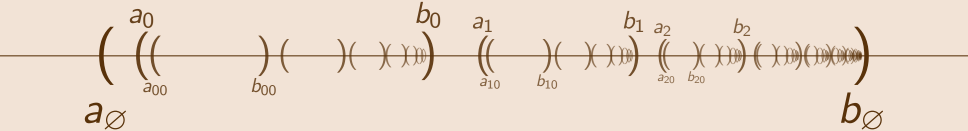
- $\varnothing$ is a short-toothed right-comb.
- X is a short-toothed right-comb, every element of X extends $\sigma \frown j$, and $i < j$, then $X \cup \{\sigma \frown i\}$ is a short-toothed right-comb.

A short-toothed right-comb



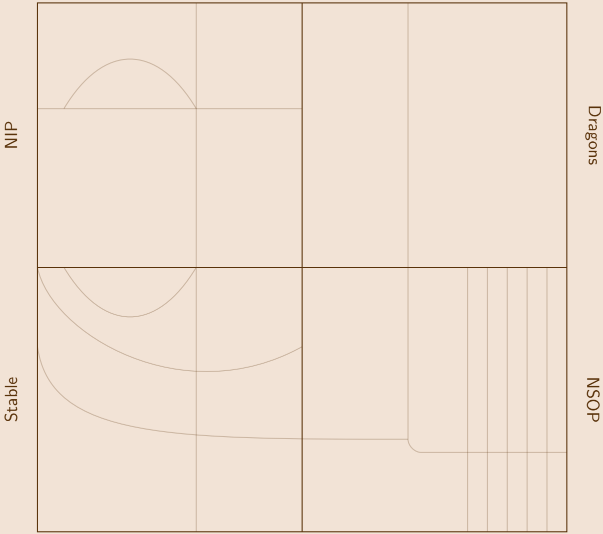
$(\mathbb{Q},<)$ has 2-SOP$_1$

In our tree in $(\mathbb{Q},<)$, any pair of incomparable elements are inconsistent.



Hence any short-toothed right-comb is 2-inconsistent.

Drawing a new line on Conant's map



Examples:

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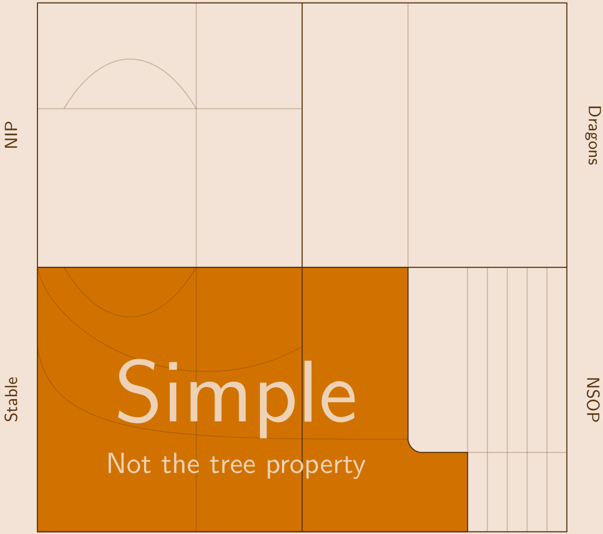

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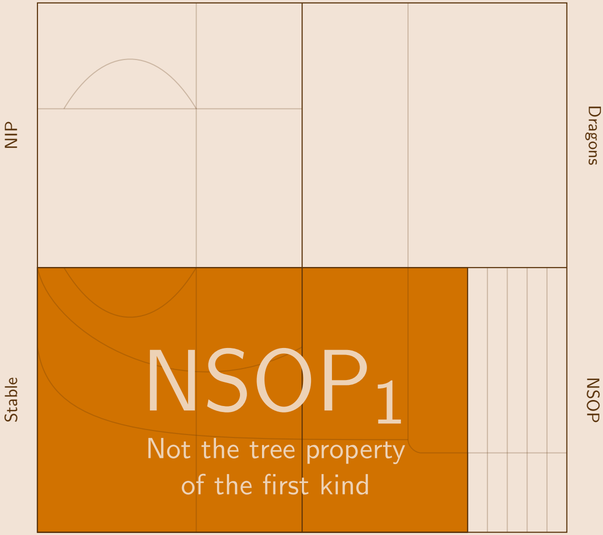

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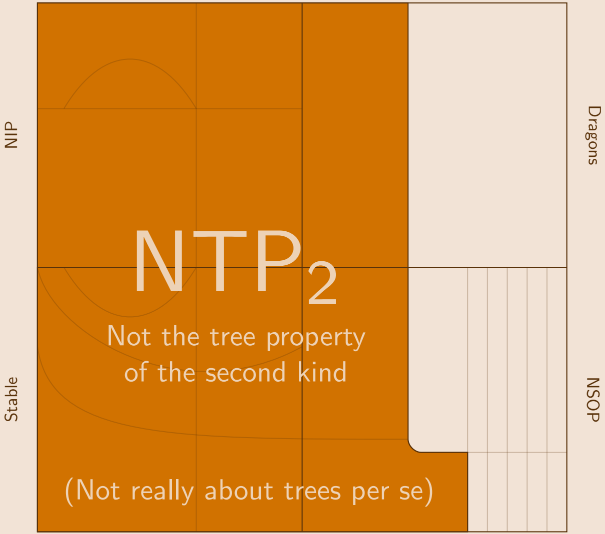

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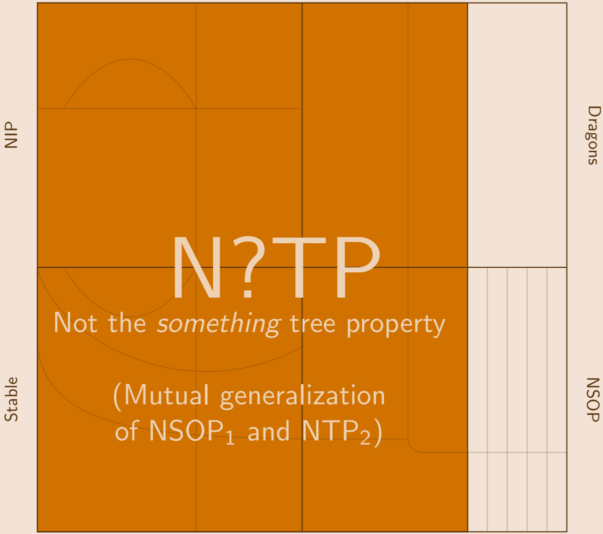

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N?TP: Generic linear order + binary function

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Given a structure M we can use an ultrafilter $\mathcal{U}$ on M (an *M-coheir*) to 'generate' a sequence of new elements (in the monster model).

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- $a_0, a_1, \ldots$ is the *Morley sequence generated by* $\mathcal{U}$.

SOP_1 in terms of coheirs

Definition

Given a coheir $\mathcal{U}$ over a model M, a formula $\varphi(x, y)$ *k-divides along* $\mathcal{U}$ if whenever $b_0, b_1, \ldots$ is a Morley sequence generated by $\mathcal{U}$, $\{\varphi(x, b_i) : i < \omega\}$ is k-inconsistent.

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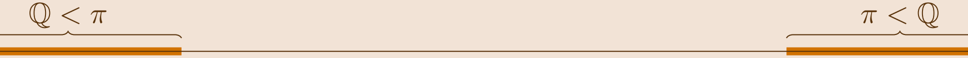
Theorem (Kaplan, Ramsey)

T has SOP_1 if and only if there is a model M, two coheirs $\mathcal{U}$ and $\mathcal{V}$ (extending the same type), and a formula $\varphi(x, y)$ such that $\varphi(x, y)$ divides along $\mathcal{U}$ but not along $\mathcal{V}$.

Coheir witnesses of SOP_1 in $(\mathbb{Q}, <)$

Two non-trivial coheirs of the 2-type living in the cut at π over $\mathbb{Q}$:



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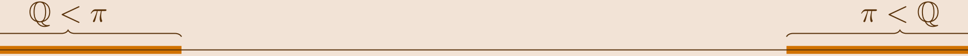

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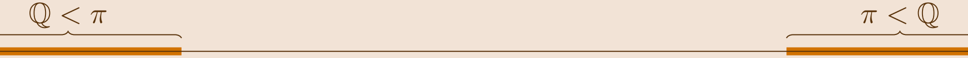

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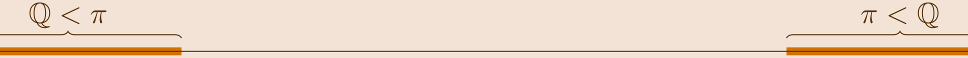

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Morley sequence generated by $\mathcal{U}_{\mathrm{pinch}}$
Consistent

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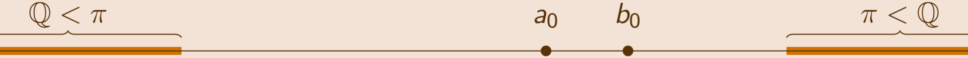

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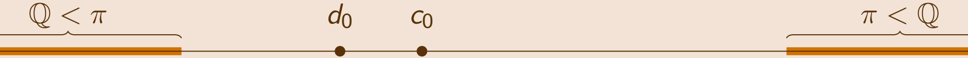

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This is non-trivial. $\mathcal{U}_{\text{pinch}}$ *does not* have this property.

TP_2 in terms of heir-coheirs

Definition

$\mathcal{U}$ is an *M-heir-coheir* if whenever b realizes $\mathcal{U}$ over $M \cup A$, there is an M-coheir $\mathcal{V}$ such that A realizes $\mathcal{V}$ over $M \cup b$.

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Theorem (Chernikov, Kaplan)

T has TP_2 if and only if there is a model M, a formula $\varphi(x, b)$, and an M-heir-coheir $\mathcal{U}$ extending the type of b over M such that $\varphi(x, b)$ divides over M but does not divide along $\mathcal{U}$.

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DLO (theory of $(\mathbb{Q}, <)$) is NTP_2.

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Lead them to the *bizarre tree property* or *BTP* (uses a weakening of heir-coheirdom).

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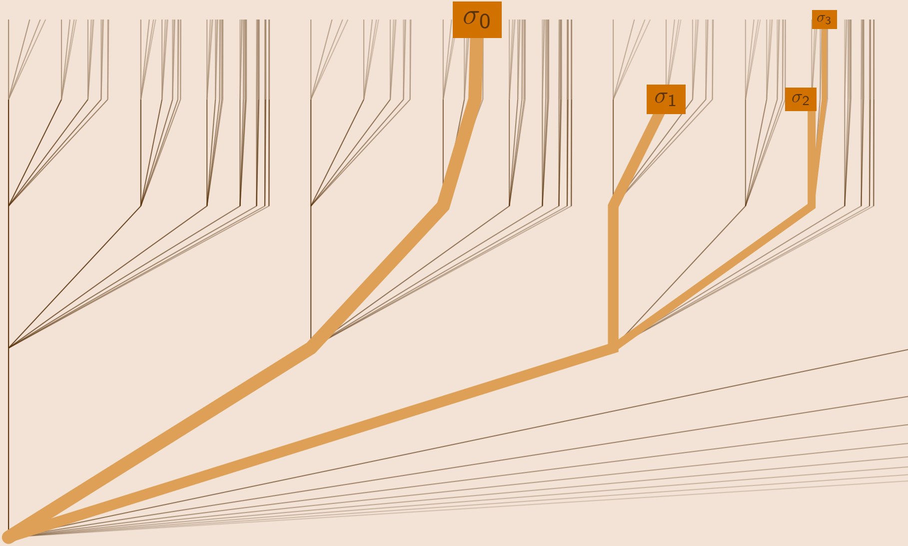
Mutchnik established the following in his proof that $\mathrm{NSOP}_1 = \mathrm{NSOP}_2$.

Theorem (Mutchnik)

The above condition without the switcheroo is equivalent to SOP_1.

A right-comb



Characterization of CTP

Theorem (H.)

A theory has k-CTP if and only if there is a model M, a formula $\varphi(x, b)$, and an M-heir-coheir $\mathcal{U}$ and an M-coheir $\mathcal{V}$ extending the type of b over M such that $\varphi(x, b)$ k-divides along $\mathcal{V}$ but does not divide along $\mathcal{U}$.

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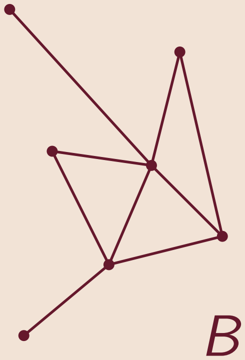
We also have the following alphabetically frustrating implication:

$$\text{ATP} \Rightarrow \text{CTP} \Rightarrow \text{BTP}$$

where the *antichain tree property* or *ATP* is another candidate for ?TP, introduced by Ahn and Kim.

What's special about heir-coheirs?

If $\mathcal{U}$ is an M-heir-coheir and B is some configuration of realizations of $\mathcal{U}$ over M, then we can find a clone B' of B with the property that every element of B' realizes $\mathcal{U}$ over $M \cup B$.



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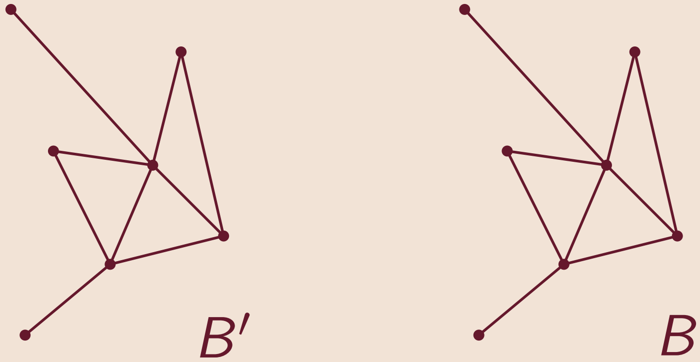

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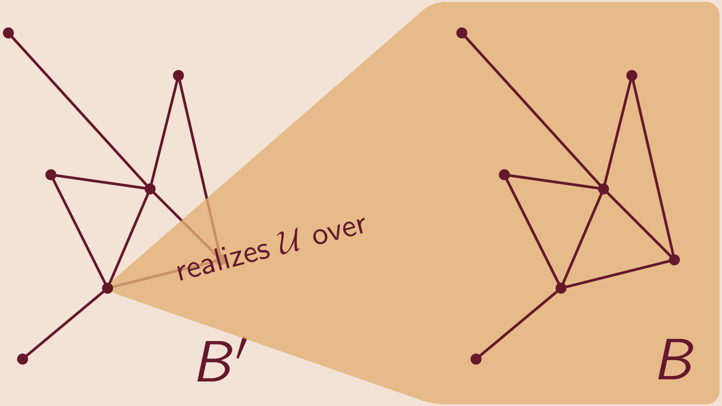

CTP from heir-coheir $\mathcal{U}$ and coheir $\mathcal{V}$

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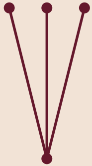

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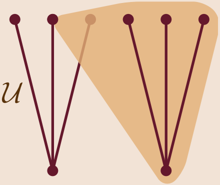

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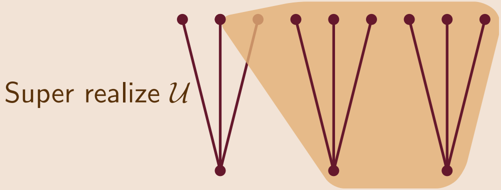

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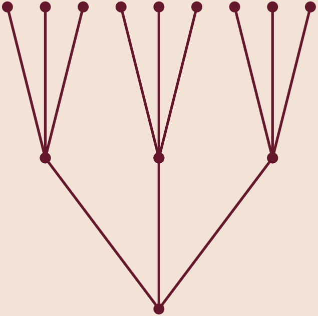

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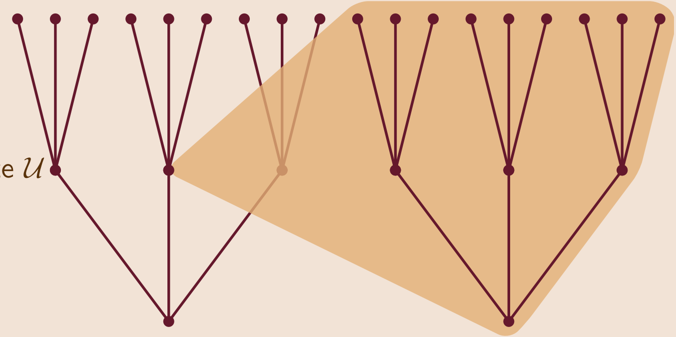
Super-duper realize $\mathcal{U}$



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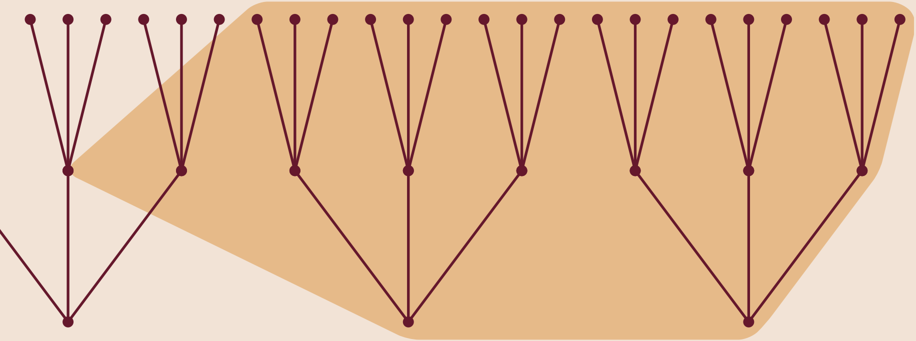

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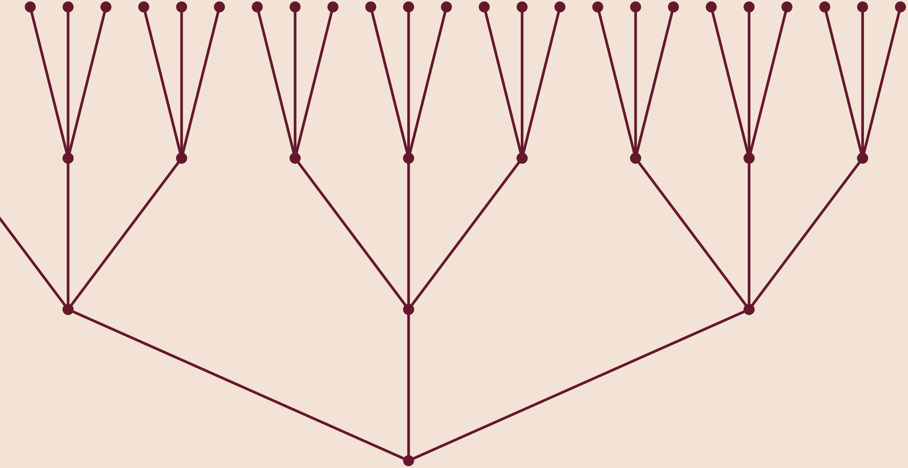

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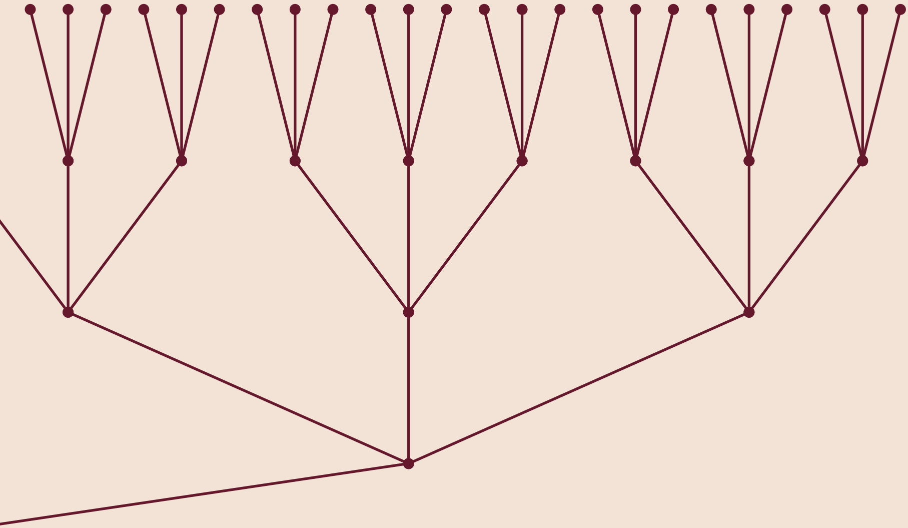

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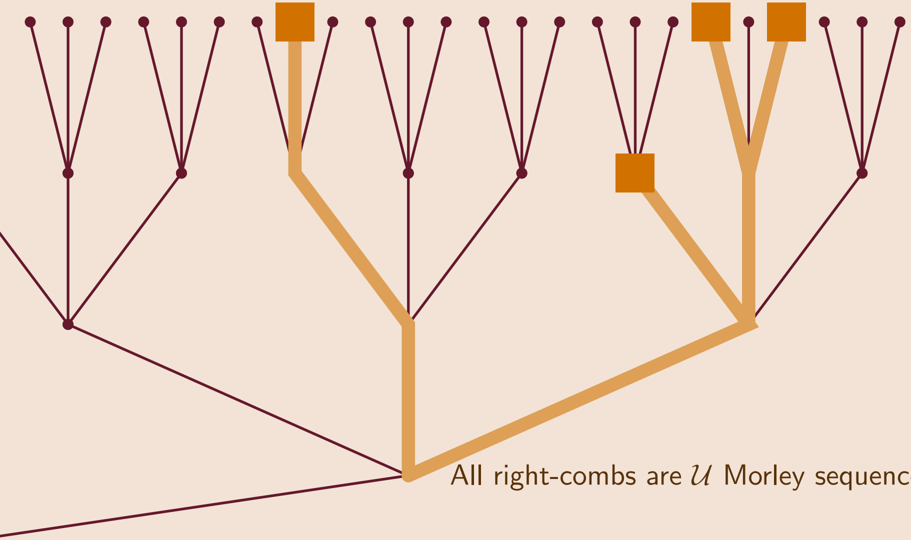

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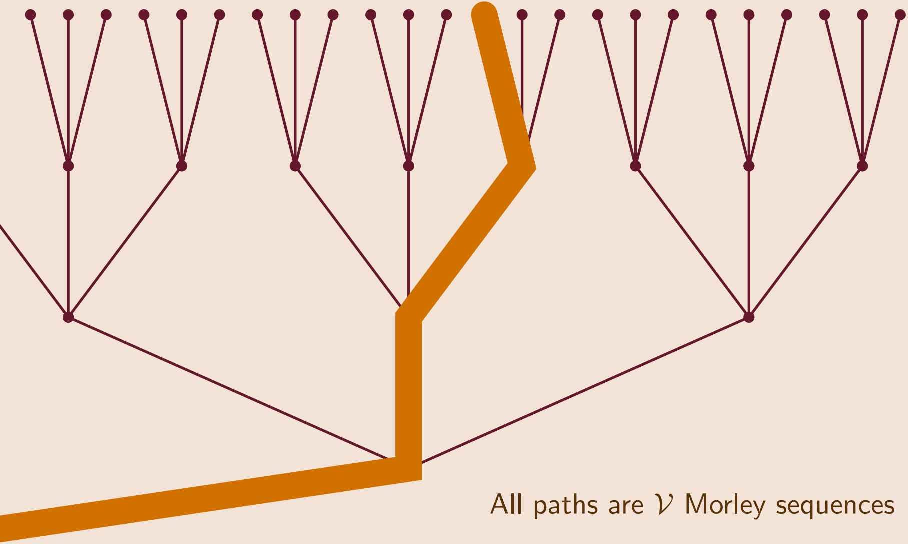

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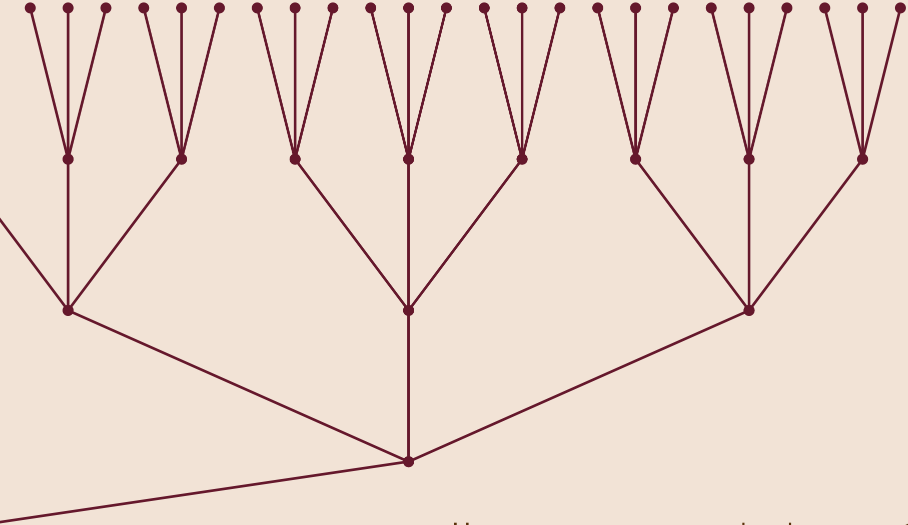

Use compactness to make the tree $\omega^{<\omega}$

Forcing

Whence do heir-coheirs come?

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There are many heir-coheirs over $(\mathbb{Q}, <)$ (any non-realized cut). Is this generalizable?

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Argue that if $\mathcal{U}$ extends the type we built and a realizes $\mathcal{U}$ over Mb, then every formula in the type of b over Ma is already realized in M by construction. □

The miniaturized argument as a blueprint for CTP

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The comb tree property (even on $2^{<\omega}$ rather than $\omega^{<\omega}$) gives you precisely what you need to generically build an heir-coheir $\mathcal{U}$ that is 'shadowed' by a coheir $\mathcal{V}$ such that the given formula divides along $\mathcal{V}$ but not along $\mathcal{U}$.

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- If $\psi(x, c)$ is an M-formula (with c in the monster) such that $\{b_\sigma : \psi(b_\sigma, c)\}$ has somewhere dense intersection with every element of $\mathcal{F}$, then there is a $d \in M$ such that $\{b_\sigma : \psi(b_\sigma, d)\} \in \mathcal{F}$.

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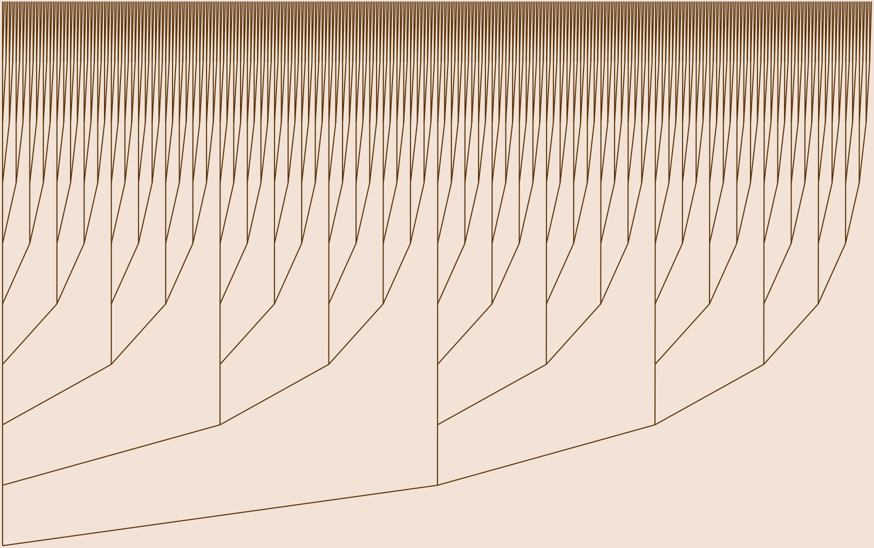
Finally, let $\mathcal{V}$ be any non-principal ultrafilter on $\{b_{\sigma_i} : i < \omega\}$. By construction, $\varphi(x, y)$ will divide along $\mathcal{V}$. Furthermore, the third bullet point will ensure that $\mathcal{U}$ and $\mathcal{V}$ extend the same type over M, so we have the required failure of Kim's lemma for coheirs and heir-coheirs.

Forcing with comb trees III



Thank you