Special coheirs and model-theoretic trees

James Hanson

Iowa State University

November 19, 2024
PKU Model Theory Seminar

Combinatorial tameness in model theory I

- Modern model theory (as of the 70s): classifying first-order theories with combinatorial tameness properties.

Combinatorial tameness in model theory I

- Modern model theory (as of the 70s): classifying first-order theories with combinatorial tameness properties.
- Started with Shelah's work generalizing Morley's theorem to uncountable languages. Ballooned into a large body of work called *stability theory*. Later extended and generalized under the title of *neostability theory*.

Combinatorial tameness in model theory I

- Modern model theory (as of the 70s): classifying first-order theories with combinatorial tameness properties.
- Started with Shelah's work generalizing Morley's theorem to uncountable languages. Ballooned into a large body of work called *stability theory*. Later extended and generalized under the title of *neostability theory*.
- There are three main adjectives that occur in stability theory: *stable*, *NIP* or *dependent*, and *NSOP*:

Combinatorial tameness in model theory I

- Modern model theory (as of the 70s): classifying first-order theories with combinatorial tameness properties.
- Started with Shelah's work generalizing Morley's theorem to uncountable languages. Ballooned into a large body of work called *stability theory*. Later extended and generalized under the title of *neostability theory*.
- There are three main adjectives that occur in stability theory: *stable*, *NIP* or *dependent*, and *NSOP*:
  - Stable: Tend to have a purely algebraic or (tame) combinatorial character: algebraically closed fields and equivalence relations.

Combinatorial tameness in model theory I

- Modern model theory (as of the 70s): classifying first-order theories with combinatorial tameness properties.
- Started with Shelah's work generalizing Morley's theorem to uncountable languages. Ballooned into a large body of work called *stability theory*. Later extended and generalized under the title of *neostability theory*.
- There are three main adjectives that occur in stability theory: *stable*, *NIP* or *dependent*, and *NSOP*:
  - Stable: Tend to have a purely algebraic or (tame) combinatorial character: algebraically closed fields and equivalence relations.
  - NIP: Tend to have a topological character: *o*-minimal theories such as real-closed fields, *p*-adic fields.

Combinatorial tameness in model theory I

- Modern model theory (as of the 70s): classifying first-order theories with combinatorial tameness properties.
- Started with Shelah's work generalizing Morley's theorem to uncountable languages. Ballooned into a large body of work called *stability theory*. Later extended and generalized under the title of *neostability theory*.
- There are three main adjectives that occur in stability theory: *stable*, *NIP* or *dependent*, and *NSOP*:
  - Stable: Tend to have a purely algebraic or (tame) combinatorial character: algebraically closed fields and equivalence relations.
  - NIP: Tend to have a topological character: *o*-minimal theories such as real-closed fields, *p*-adic fields.
  - NSOP: Doesn't have much of a structure theory, but there is an important subclass of *simple* theories, which are typically thought of as 'stable + random noise':

Combinatorial tameness in model theory I

- Modern model theory (as of the 70s): classifying first-order theories with combinatorial tameness properties.
- Started with Shelah's work generalizing Morley's theorem to uncountable languages. Ballooned into a large body of work called *stability theory*. Later extended and generalized under the title of *neostability theory*.
- There are three main adjectives that occur in stability theory: *stable*, *NIP* or *dependent*, and *NSOP*:
  - Stable: Tend to have a purely algebraic or (tame) combinatorial character: algebraically closed fields and equivalence relations.
  - NIP: Tend to have a topological character: *o*-minimal theories such as real-closed fields, *p*-adic fields.
  - NSOP: Doesn't have much of a structure theory, but there is an important subclass of *simple* theories, which are typically thought of as 'stable + random noise': pseudo-finite fields and the random graph.

Combinatorial tameness in model theory II

- Important relationship between these notions:

$$\text{Stable} = \text{NIP} \cap \text{Simple} = \text{NIP} \cap \text{NSOP}$$

Combinatorial tameness in model theory II

- Important relationship between these notions:

$$\text{Stable} = \text{NIP} \cap \text{Simple} = \text{NIP} \cap \text{NSOP}$$

- A lot of these adjectives are defined in terms of combinatorial consistency patterns, often involving trees.

Combinatorial tameness in model theory II

- Important relationship between these notions:

$$\text{Stable} = \text{NIP} \cap \text{Simple} = \text{NIP} \cap \text{NSOP}$$

- A lot of these adjectives are defined in terms of combinatorial consistency patterns, often involving trees.
- One common endeavor in model theory is trying to find new adjectives with tractable structure theory.

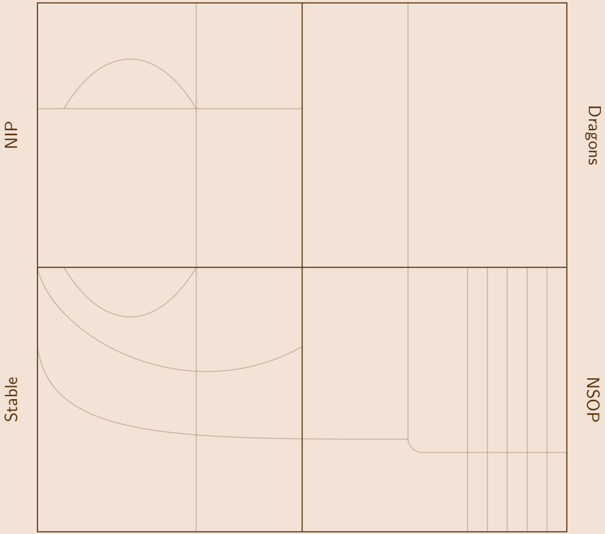
Combinatorial tameness in model theory II

- Important relationship between these notions:

$$\text{Stable} = \text{NIP} \cap \text{Simple} = \text{NIP} \cap \text{NSOP}$$

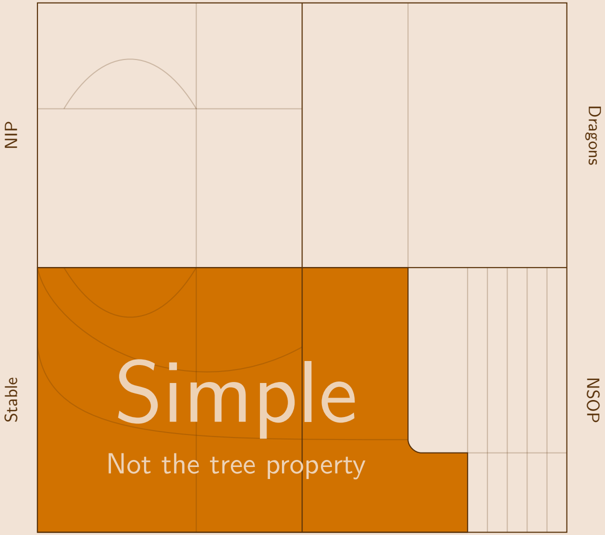
- A lot of these adjectives are defined in terms of combinatorial consistency patterns, often involving trees.
- One common endeavor in model theory is trying to find new adjectives with tractable structure theory. Example: NTP_2 is the 'least common generalization' of NIP and simplicity.

The map



Examples:

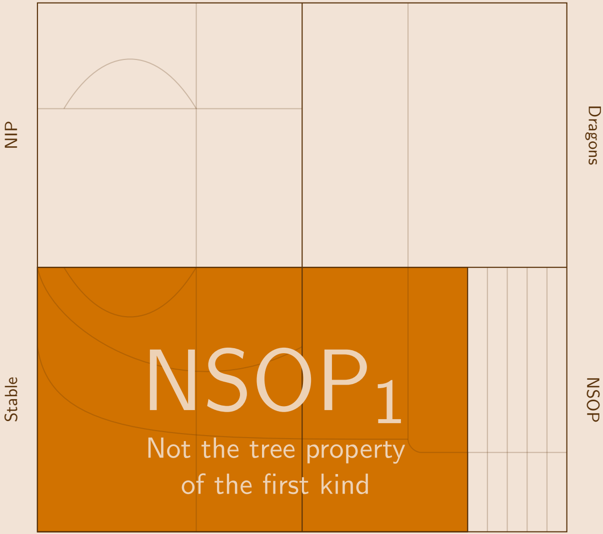
The map



Examples:

Simple: Generic graph

The map



Examples:

Simple: Generic graph

NSOP$_1$: Generic binary function

The map

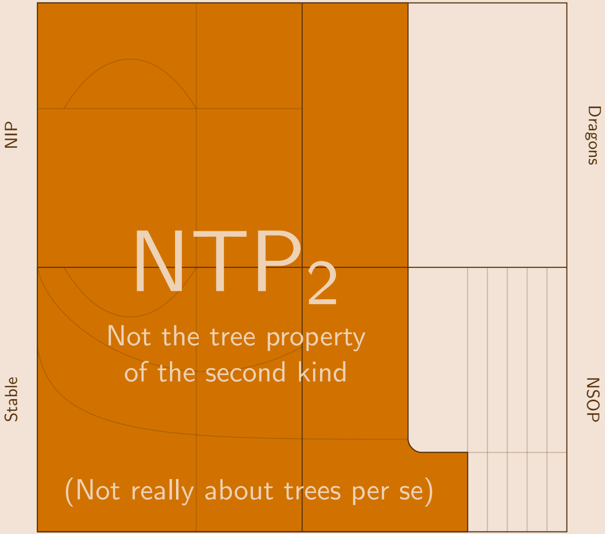
Examples:

Simple: Generic graph

$NSOP_1$: Generic binary function

NTP_2: Generic linearly ordered graph

The map



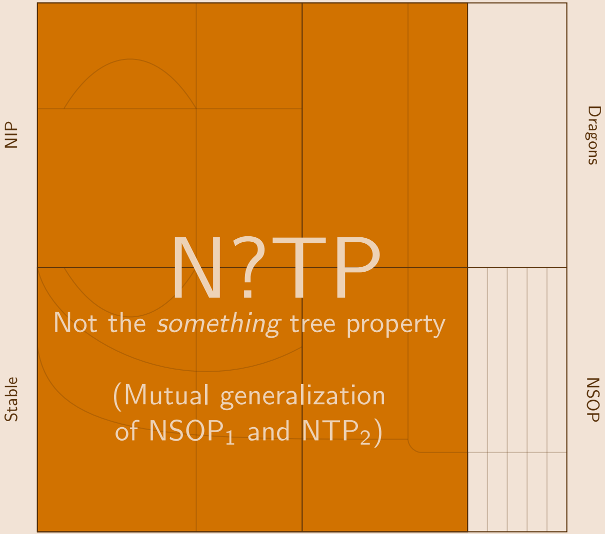
Examples:

Simple: Generic graph

NSOP_1: Generic binary function

NTP_2: Generic linearly ordered graph

The map



Examples:

Simple: Generic graph

$NSOP_1$: Generic binary function

NTP_2: Generic linearly ordered graph

N?TP: Generic linear order + binary function

The tree property in model theory

A formula $\varphi(x, y)$ has the *k-tree property* if there is a tree $(c_\sigma)_{\sigma \in \omega^{<\omega}}$ of parameters such that

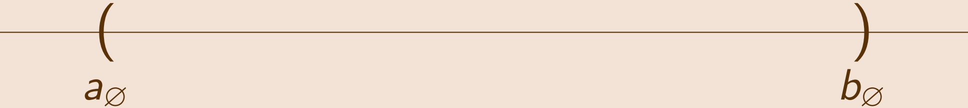
- paths are consistent: $\{\varphi(x, c_{\alpha \upharpoonright n}) : n < \omega\}$ for $\alpha \in \omega^\omega$,
- siblings are k-inconsistent: $\{\varphi(x, c_{\sigma \frown n}) : n < \omega\}$.

The tree property in model theory

A formula $\varphi(x, y)$ has the *k-tree property* if there is a tree $(c_\sigma)_{\sigma \in \omega^{<\omega}}$ of parameters such that

- paths are consistent: $\{\varphi(x, c_{\alpha \upharpoonright n}) : n < \omega\}$ for $\alpha \in \omega^\omega$,
- siblings are k-inconsistent: $\{\varphi(x, c_{\sigma \frown n}) : n < \omega\}$.

Example $\varphi(x, a, b) = (a < x < b)$ with $c = ab$ in $(\mathbb{Q}, <)$:

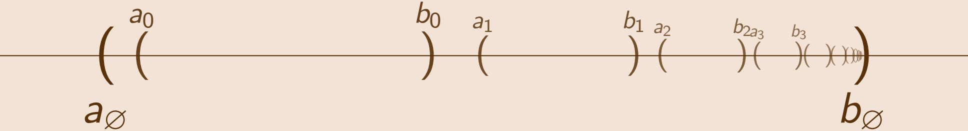


The tree property in model theory

A formula $\varphi(x, y)$ has the *k-tree property* if there is a tree $(c_\sigma)_{\sigma \in \omega^{<\omega}}$ of parameters such that

- paths are consistent: $\{\varphi(x, c_{\alpha \restriction n}) : n < \omega\}$ for $\alpha \in \omega^\omega$,
- siblings are k-inconsistent: $\{\varphi(x, c_{\sigma \frown n}) : n < \omega\}$.

Example $\varphi(x, a, b) = (a < x < b)$ with $c = ab$ in $(\mathbb{Q}, <)$:

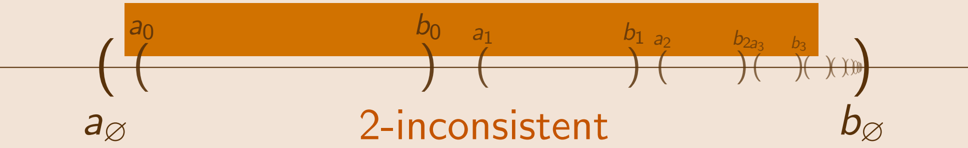


The tree property in model theory

A formula $\varphi(x, y)$ has the *k-tree property* if there is a tree $(c_\sigma)_{\sigma \in \omega^{<\omega}}$ of parameters such that

- paths are consistent: $\{\varphi(x, c_{\alpha\restriction n}) : n < \omega\}$ for $\alpha \in \omega^\omega$,
- siblings are k-inconsistent: $\{\varphi(x, c_{\sigma\frown n}) : n < \omega\}$.

Example $\varphi(x, a, b) = (a < x < b)$ with $c = ab$ in $(\mathbb{Q}, <)$:

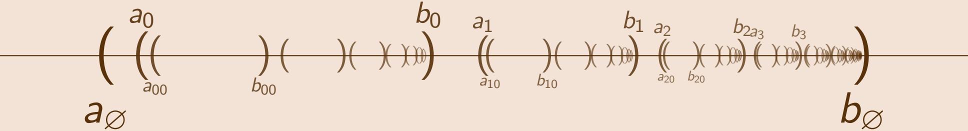


The tree property in model theory

A formula $\varphi(x, y)$ has the *k-tree property* if there is a tree $(c_\sigma)_{\sigma \in \omega^{<\omega}}$ of parameters such that

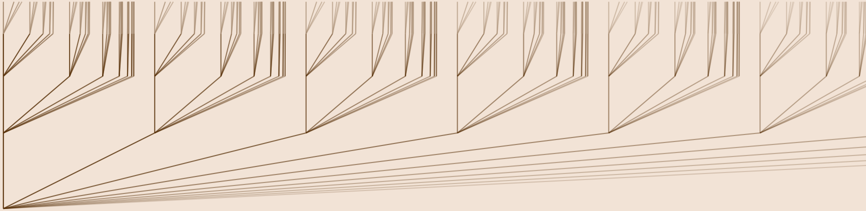
- paths are consistent: $\{\varphi(x, c_{\alpha \restriction n}) : n < \omega\}$ for $\alpha \in \omega^\omega$,
- siblings are k-inconsistent: $\{\varphi(x, c_{\sigma \frown n}) : n < \omega\}$.

Example $\varphi(x, a, b) = (a < x < b)$ with $c = ab$ in $(\mathbb{Q}, <)$:



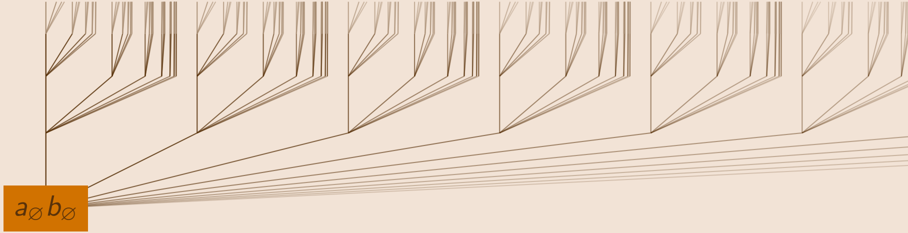
The tree in the tree property

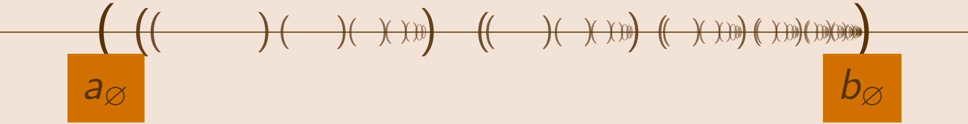
$\omega^{<\omega}$



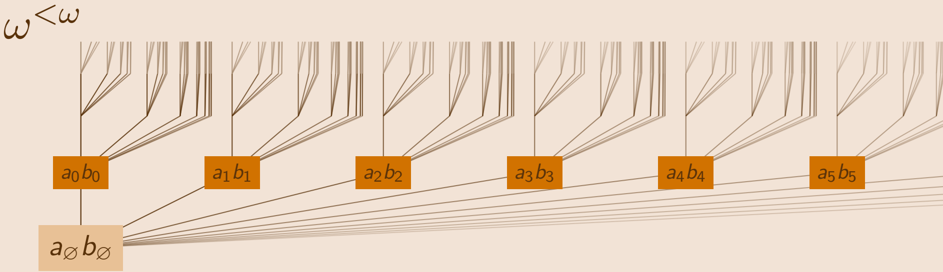
The tree in the tree property

$\omega^{<\omega}$

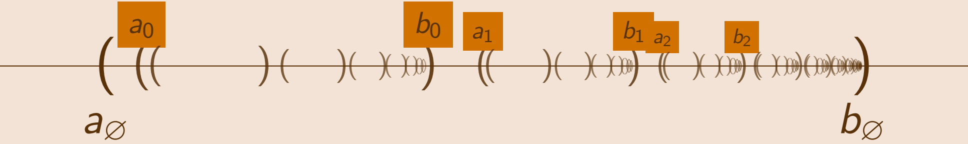




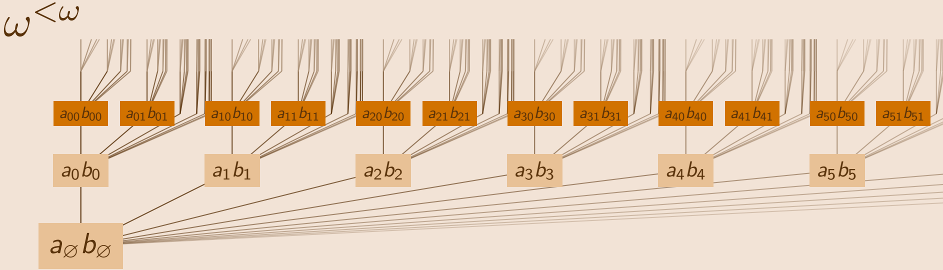
The tree in the tree property



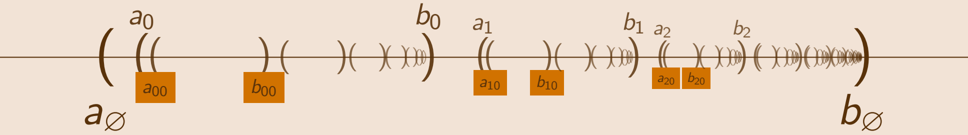
Siblings are 2-inconsistent



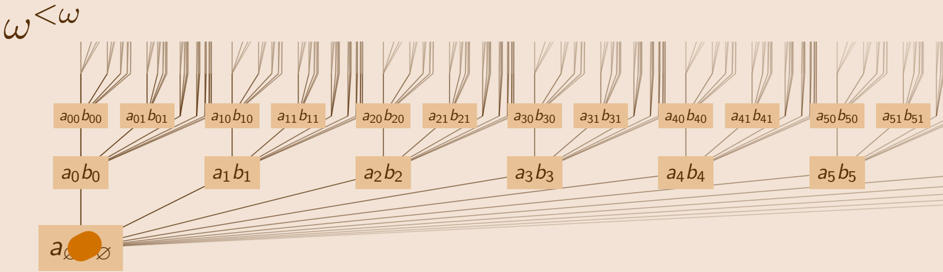
The tree in the tree property



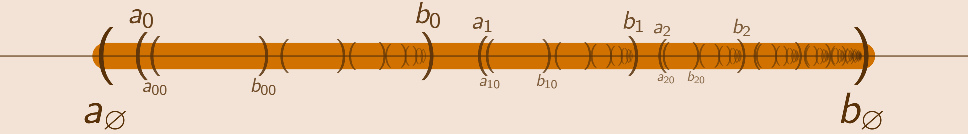
Siblings are 2-inconsistent



The tree in the tree property

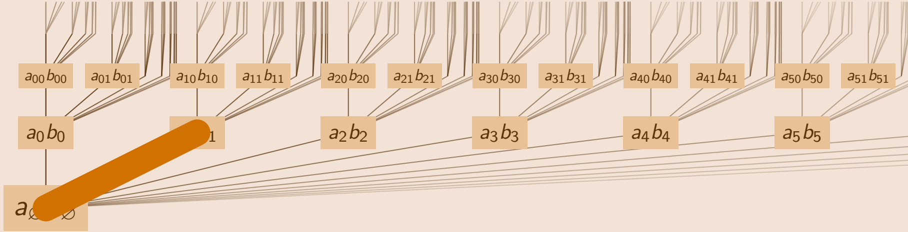


Paths are consistent

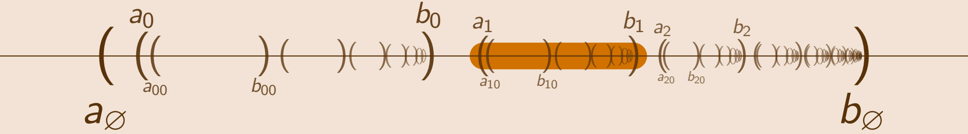


The tree in the tree property

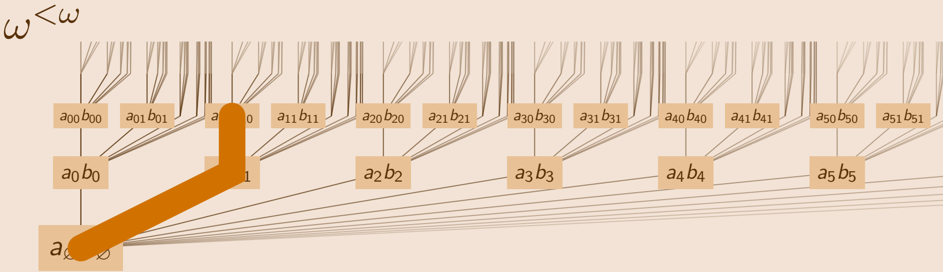
$\omega^{<\omega}$



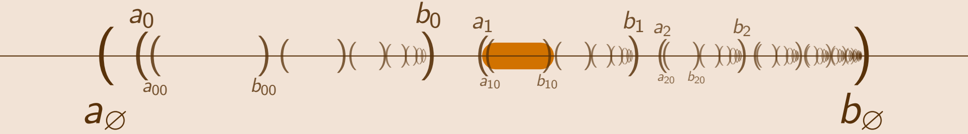
Paths are consistent



The tree in the tree property



Paths are consistent



The tree property of the first kind

(Extremely revisionist definition) A formula $\varphi(x, c)$ has the *k-tree property of the first kind*

The tree property of the first kind

(Extremely revisionist definition) A formula $\varphi(x, c)$ has the *k-tree property of the first kind* or *k-SOP*$_1$

The tree property of the first kind

(Extremely revisionist definition) A formula $\varphi(x, c)$ has the *k-tree property of the first kind* or *k-SOP_1* if there is a tree $(c_\sigma)_{\sigma \in \omega^{<\omega}}$ of parameters such that

The tree property of the first kind

(Extremely revisionist definition) A formula $\varphi(x, c)$ has the *k-tree property of the first kind* or *k-SOP$_1$* if there is a tree $(c_\sigma)_{\sigma \in \omega^{<\omega}}$ of parameters such that

- paths are consistent: $\{\varphi(x, c_{\alpha \restriction n}) : n < \omega\}$ for $\alpha \in \omega^\omega$,

The tree property of the first kind

(Extremely revisionist definition) A formula $\varphi(x, c)$ has the *k-tree property of the first kind* or *k-SOP$_1$* if there is a tree $(c_\sigma)_{\sigma \in \omega^{<\omega}}$ of parameters such that

- paths are consistent: $\{\varphi(x, c_{\alpha \upharpoonright n}) : n < \omega\}$ for $\alpha \in \omega^\omega$,
- for any short-toothed right-comb $X \subset \omega^{<\omega}$, $\{\varphi(x, c_\sigma) : \sigma \in X\}$ is k-inconsistent.

The tree property of the first kind

(Extremely revisionist definition) A formula $\varphi(x, c)$ has the *k-tree property of the first kind* or *k-SOP$_1$* if there is a tree $(c_\sigma)_{\sigma \in \omega^{<\omega}}$ of parameters such that

- paths are consistent: $\{\varphi(x, c_{\alpha \upharpoonright n}) : n < \omega\}$ for $\alpha \in \omega^\omega$,
- for any short-toothed right-comb $X \subset \omega^{<\omega}$, $\{\varphi(x, c_\sigma) : \sigma \in X\}$ is k-inconsistent.

Short-toothed right-combs are defined inductively:

The tree property of the first kind

(Extremely revisionist definition) A formula $\varphi(x, c)$ has the *k-tree property of the first kind* or *k-SOP$_1$* if there is a tree $(c_\sigma)_{\sigma \in \omega^{<\omega}}$ of parameters such that

- paths are consistent: $\{\varphi(x, c_{\alpha \upharpoonright n}) : n < \omega\}$ for $\alpha \in \omega^\omega$,
- for any short-toothed right-comb $X \subset \omega^{<\omega}$, $\{\varphi(x, c_\sigma) : \sigma \in X\}$ is k-inconsistent.

Short-toothed right-combs are defined inductively:

- $\varnothing$ is a short-toothed right-comb.

The tree property of the first kind

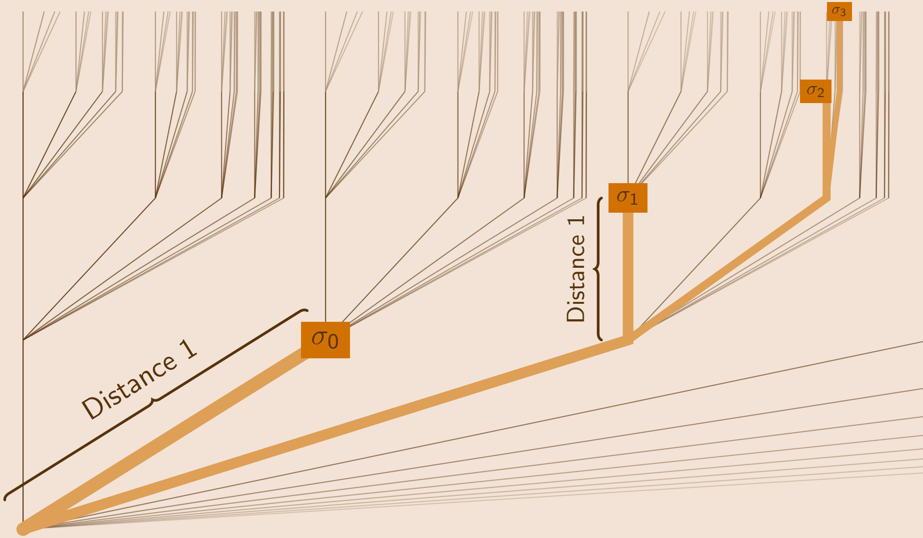
(Extremely revisionist definition) A formula $\varphi(x, c)$ has the *k-tree property of the first kind* or *k-SOP$_1$* if there is a tree $(c_\sigma)_{\sigma \in \omega^{<\omega}}$ of parameters such that

- paths are consistent: $\{\varphi(x, c_{\alpha \restriction n}) : n < \omega\}$ for $\alpha \in \omega^\omega$,
- for any short-toothed right-comb $X \subset \omega^{<\omega}$, $\{\varphi(x, c_\sigma) : \sigma \in X\}$ is k-inconsistent.

Short-toothed right-combs are defined inductively:

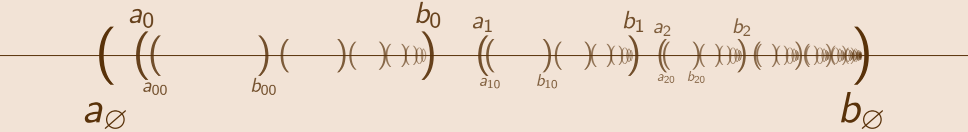
- $\varnothing$ is a short-toothed right-comb.
- X is a short-toothed right-comb, every element of X extends $\sigma \frown j$, and $i < j$, then $X \cup \{\sigma \frown i\}$ is a short-toothed right-comb.

A short-toothed right-comb



$(\mathbb{Q}, <)$ has 2-SOP_1

In our tree in $(\mathbb{Q}, <)$, any pair of incomparable elements are inconsistent.



Hence any short-toothed right-comb is 2-inconsistent.

Coheirs

Coheirs

Given a structure M we can use an ultrafilter $\mathcal{U}$ on M (an *M-coheir*) to 'generate' a sequence of new elements (in the monster model).

Coheirs

Given a structure M we can use an ultrafilter $\mathcal{U}$ on M (an *M-coheir*) to 'generate' a sequence of new elements (in the monster model).

Example $(\mathbb{Q}, <)$ with ultrafilter concentrating at $+\infty$:

Coheirs

Given a structure M we can use an ultrafilter $\mathcal{U}$ on M (an *M-coheir*) to 'generate' a sequence of new elements (in the monster model).

Example $(\mathbb{Q}, <)$ with ultrafilter concentrating at $+\infty$:

Coheirs

Given a structure M we can use an ultrafilter $\mathcal{U}$ on M (an *M-coheir*) to 'generate' a sequence of new elements (in the monster model).

Example $(\mathbb{Q}, <)$ with ultrafilter concentrating at $+\infty$:

Coheirs

Given a structure M we can use an ultrafilter $\mathcal{U}$ on M (an *M-coheir*) to 'generate' a sequence of new elements (in the monster model).

Example $(\mathbb{Q}, <)$ with ultrafilter concentrating at $+\infty$:

- a_{i+1} is what $\mathcal{U}$ 'looks like' to $\mathbb{Q}$ and $a_0, \ldots, a_i$.

Coheirs

Given a structure M we can use an ultrafilter $\mathcal{U}$ on M (an *M-coheir*) to 'generate' a sequence of new elements (in the monster model).

Example $(\mathbb{Q}, <)$ with ultrafilter concentrating at $+\infty$:

- a_{i+1} is what $\mathcal{U}$ 'looks like' to $\mathbb{Q}$ and $a_0, \dots, a_i$.
- a_{i+1} *realizes* $\mathcal{U}$ *over* $\mathbb{Q} \cup \{a_0, \dots, a_i\}$.

Coheirs

Given a structure M we can use an ultrafilter $\mathcal{U}$ on M (an *M-coheir*) to 'generate' a sequence of new elements (in the monster model).

Example $(\mathbb{Q}, <)$ with ultrafilter concentrating at $+\infty$:

- a_{i+1} is what $\mathcal{U}$ 'looks like' to $\mathbb{Q}$ and $a_0, \ldots, a_i$.
- a_{i+1} *realizes* $\mathcal{U}$ *over* $\mathbb{Q} \cup \{a_0, \ldots, a_i\}$.
- $a_0, a_1, \ldots$ is the *Morley sequence generated by* $\mathcal{U}$.

SOP_1 in terms of coheirs

Definition

Given a coheir $\mathcal{U}$ over a model M, a formula $\varphi(x, y)$ *k-divides along* $\mathcal{U}$ if whenever $b_0, b_1, \ldots$ is a Morley sequence generated by $\mathcal{U}$, $\{\varphi(x, b_i) : i < \omega\}$ is k-inconsistent.

SOP_1 in terms of coheirs

Definition

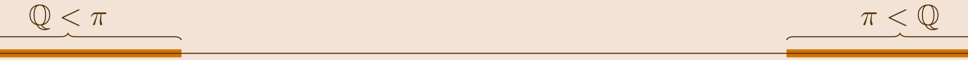
Given a coheir $\mathcal{U}$ over a model M, a formula $\varphi(x, y)$ *k-divides along* $\mathcal{U}$ if whenever $b_0, b_1, \ldots$ is a Morley sequence generated by $\mathcal{U}$, $\{\varphi(x, b_i) : i < \omega\}$ is k-inconsistent.

Theorem (Kaplan, Ramsey)

T has SOP_1 if and only if there is a model M, two coheirs $\mathcal{U}$ and $\mathcal{V}$ (extending the same type), and a formula $\varphi(x, y)$ such that $\varphi(x, y)$ divides along $\mathcal{U}$ but not along $\mathcal{V}$.

Coheir witnesses of SOP_1 in $(\mathbb{Q}, <)$

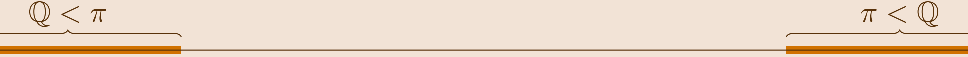
Two non-trivial coheirs of the 2-type living in the cut at π over $\mathbb{Q}$:



Coheir witnesses of SOP_1 in $(\mathbb{Q}, <)$

Two non-trivial coheirs of the 2-type living in the cut at π over $\mathbb{Q}$:

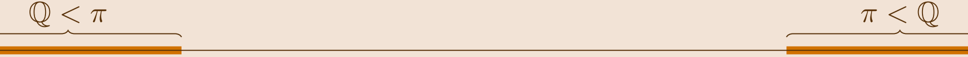
- $\mathcal{U}_{\mathsf{pinch}}$ corresponding to two elements 'pinching' the cut (coming in from both sides).



Coheir witnesses of SOP_1 in $(\mathbb{Q}, <)$

Two non-trivial coheirs of the 2-type living in the cut at π over $\mathbb{Q}$:

- $\mathcal{U}_{\mathsf{pinch}}$ corresponding to two elements 'pinching' the cut (coming in from both sides).
- $\mathcal{U}_{\mathsf{below}}$ corresponding to two elements sliding towards the cut from below.

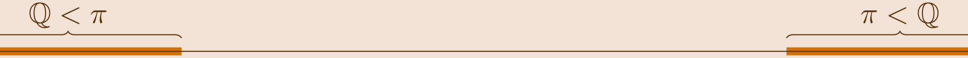


Coheir witnesses of SOP_1 in $(\mathbb{Q}, <)$

Two non-trivial coheirs of the 2-type living in the cut at π over $\mathbb{Q}$:

- $\mathcal{U}_{\mathsf{pinch}}$ corresponding to two elements 'pinching' the cut (coming in from both sides).
- $\mathcal{U}_{\mathsf{below}}$ corresponding to two elements sliding towards the cut from below.

The formula $(a < x < b)$ divides along $\mathcal{U}_{\mathsf{below}}$ but not along $\mathcal{U}_{\mathsf{pinch}}$.



Coheir witnesses of SOP_1 in $(\mathbb{Q}, <)$

Two non-trivial coheirs of the 2-type living in the cut at π over $\mathbb{Q}$:

- $\mathcal{U}_{\mathsf{pinch}}$ corresponding to two elements 'pinching' the cut (coming in from both sides).
- $\mathcal{U}_{\mathsf{below}}$ corresponding to two elements sliding towards the cut from below.

The formula $(a < x < b)$ divides along $\mathcal{U}_{\mathsf{below}}$ but not along $\mathcal{U}_{\mathsf{pinch}}$.

Morley sequence generated by $\mathcal{U}_{\mathsf{pinch}}$
Consistent

Coheir witnesses of SOP_1 in $(\mathbb{Q}, <)$

Two non-trivial coheirs of the 2-type living in the cut at π over $\mathbb{Q}$:

- $\mathcal{U}_{\text{pinch}}$ corresponding to two elements 'pinching' the cut (coming in from both sides).
- $\mathcal{U}_{\text{below}}$ corresponding to two elements sliding towards the cut from below.

The formula $(a < x < b)$ divides along $\mathcal{U}_{\text{below}}$ but not along $\mathcal{U}_{\text{pinch}}$.

Morley sequence generated by $\mathcal{U}_{\text{below}}$
2-inconsistent

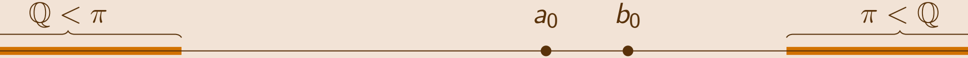
Heir-coheirs

$\mathcal{U}_{\text{below}}$ has a special property. The Morley sequence it generates

Generated by $\mathcal{U}_{\text{below}}$

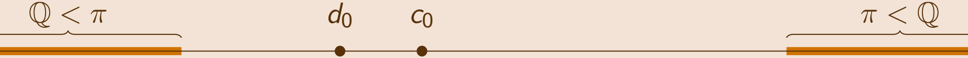
Heir-coheirs

$\mathcal{U}_{\text{below}}$ has a special property. The Morley sequence it generates



Generated by $\mathcal{U}_{\text{below}}$

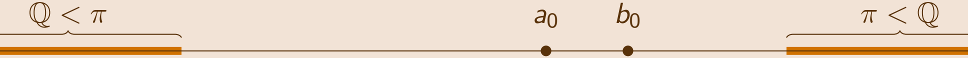
is 'the same' as the Morley sequence generated by a different coheir backwards:



Generated by $\mathcal{U}_{\text{above}}$

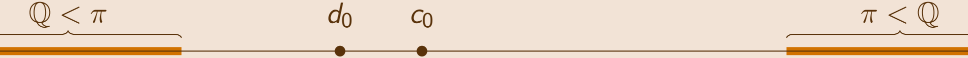
Heir-coheirs

$\mathcal{U}_{\text{below}}$ has a special property. The Morley sequence it generates



Generated by $\mathcal{U}_{\text{below}}$

is 'the same' as the Morley sequence generated by a different coheir backwards:



Generated by $\mathcal{U}_{\text{above}}$

This is non-trivial. $\mathcal{U}_{\text{pinch}}$ *does not* have this property.

TP_2 in terms of heir-coheirs

Definition

$\mathcal{U}$ is an *M-heir-coheir* if whenever b realizes $\mathcal{U}$ over $M \cup A$, there is an M-coheir $\mathcal{V}$ such that A realizes $\mathcal{V}$ over $M \cup b$.

TP_2 in terms of heir-coheirs

Definition

$\mathcal{U}$ is an *M-heir-coheir* if whenever b realizes $\mathcal{U}$ over $M \cup A$, there is an M-coheir $\mathcal{V}$ such that A realizes $\mathcal{V}$ over $M \cup b$.

A formula $\varphi(x, b)$ *k-divides over M* if there is a sequence $(b_i)_{i<\omega}$ of realizations of the type of b over M such that $\{\varphi(x, b_i) : i < \omega\}$ is k-inconsistent.

TP_2 in terms of heir-coheirs

Definition

$\mathcal{U}$ is an *M-heir-coheir* if whenever b realizes $\mathcal{U}$ over $M \cup A$, there is an M-coheir $\mathcal{V}$ such that A realizes $\mathcal{V}$ over $M \cup b$.

A formula $\varphi(x, b)$ *k-divides over M* if there is a sequence $(b_i)_{i<\omega}$ of realizations of the type of b over M such that $\{\varphi(x, b_i) : i < \omega\}$ is k-inconsistent.

Theorem (Chernikov, Kaplan)

T has TP_2 if and only if there is a model M, a formula $\varphi(x, b)$, and an M-heir-coheir $\mathcal{U}$ extending the type of b over M such that $\varphi(x, b)$ divides over M but does not divide along $\mathcal{U}$.

TP_2 in terms of heir-coheirs

Definition

$\mathcal{U}$ is an *M-heir-coheir* if whenever b realizes $\mathcal{U}$ over $M \cup A$, there is an M-coheir $\mathcal{V}$ such that A realizes $\mathcal{V}$ over $M \cup b$.

A formula $\varphi(x, b)$ *k-divides over M* if there is a sequence $(b_i)_{i<\omega}$ of realizations of the type of b over M such that $\{\varphi(x, b_i) : i < \omega\}$ is k-inconsistent.

Theorem (Chernikov, Kaplan)

T has TP_2 if and only if there is a model M, a formula $\varphi(x, b)$, and an M-heir-coheir $\mathcal{U}$ extending the type of b over M such that $\varphi(x, b)$ divides over M but does not divide along $\mathcal{U}$.

DLO (theory of $(\mathbb{Q}, <)$) is NTP_2.

N?TP via a new Kim's lemma?

Kruckman and Ramsey suggested formulating N?TP via a mutual generalization of the Kim's lemmas for NSOP_1 and NTP_2.

N?TP via a new Kim's lemma?

Kruckman and Ramsey suggested formulating N?TP via a mutual generalization of the Kim's lemmas for NSOP_1 and NTP_2.

- NSOP_1: If $\varphi(x, b)$ divides along some coheir, then it divides along every coheir.

N?TP via a new Kim's lemma?

Kruckman and Ramsey suggested formulating N?TP via a mutual generalization of the Kim's lemmas for NSOP_1 and NTP_2.

- NSOP_1: If $\varphi(x, b)$ divides along some coheir, then it divides along every coheir.
- NTP_2: If $\varphi(x, b)$ divides, then it divides along every heir-coheir.

N?TP via a new Kim's lemma?

Kruckman and Ramsey suggested formulating N?TP via a mutual generalization of the Kim's lemmas for $NSOP_1$ and NTP_2.

- $NSOP_1$: If $\varphi(x, b)$ divides along some coheir, then it divides along every coheir.
- NTP_2: If $\varphi(x, b)$ divides, then it divides along every heir-coheir.

Lead them to the *bizarre tree property* or *BTP* (uses a weakening of heir-coheirdom).

N?TP via a new Kim's lemma?

Kruckman and Ramsey suggested formulating N?TP via a mutual generalization of the Kim's lemmas for NSOP_1 and NTP_2.

- NSOP_1: If $\varphi(x, b)$ divides along some coheir, then it divides along every coheir.
- NTP_2: If $\varphi(x, b)$ divides, then it divides along every heir-coheir.

Lead them to the *bizarre tree property* or *BTP* (uses a weakening of heir-coheirdom).

Their philosophy also suggests the following:

N?TP via a new Kim's lemma?

Kruckman and Ramsey suggested formulating N?TP via a mutual generalization of the Kim's lemmas for $NSOP_1$ and NTP_2.

- $NSOP_1$: If $\varphi(x, b)$ divides along some coheir, then it divides along every coheir.
- NTP_2: If $\varphi(x, b)$ divides, then it divides along every heir-coheir.

Lead them to the *bizarre tree property* or *BTP* (uses a weakening of heir-coheirdom).

Their philosophy also suggests the following:

? N?TP: If $\varphi(x, b)$ divides along some coheir, then it divides along every heir-coheir?

Combs

Combs

A formula $\varphi(x, c)$ has the *k-comb tree property* or *k-CTP* if there is a tree $(c_\sigma)_{\sigma \in \omega^{<\omega}}$ of parameters such that

Combs

A formula $\varphi(x, c)$ has the *k-comb tree property* or *k-CTP* if there is a tree $(c_\sigma)_{\sigma \in \omega^{<\omega}}$ of parameters such that

- paths are k-**inconsistent**: $\{\varphi(x, c_{\alpha \restriction n}) : n < \omega\}$ for $\alpha \in \omega^\omega$,

Combs

A formula $\varphi(x, c)$ has the *k-comb tree property* or *k-CTP* if there is a tree $(c_\sigma)_{\sigma \in \omega^{<\omega}}$ of parameters such that

- paths are k-**inconsistent**: $\{\varphi(x, c_{\alpha \upharpoonright n}) : n < \omega\}$ for $\alpha \in \omega^\omega$,
- for any right-comb $X \subset \omega^{<\omega}$, $\{\varphi(x, c_\sigma) : \sigma \in X\}$ is **consistent**.

(Note the switcheroo.)

Combs

A formula $\varphi(x, c)$ has the *k-comb tree property* or *k-CTP* if there is a tree $(c_\sigma)_{\sigma \in \omega^{<\omega}}$ of parameters such that

- paths are k-**inconsistent**: $\{\varphi(x, c_{\alpha \upharpoonright n}) : n < \omega\}$ for $\alpha \in \omega^\omega$,
- for any right-comb $X \subset \omega^{<\omega}$, $\{\varphi(x, c_\sigma) : \sigma \in X\}$ is **consistent**.

(Note the switcheroo.)
Right-combs are defined inductively:

Combs

A formula $\varphi(x, c)$ has the *k-comb tree property* or *k-CTP* if there is a tree $(c_\sigma)_{\sigma \in \omega^{<\omega}}$ of parameters such that

- paths are k-**inconsistent**: $\{\varphi(x, c_{\alpha \restriction n}) : n < \omega\}$ for $\alpha \in \omega^\omega$,
- for any right-comb $X \subset \omega^{<\omega}$, $\{\varphi(x, c_\sigma) : \sigma \in X\}$ is **consistent**.

(Note the switcheroo.)
Right-combs are defined inductively:

- $\varnothing$ is a right-comb.

Combs

A formula $\varphi(x, c)$ has the *k-comb tree property* or *k-CTP* if there is a tree $(c_\sigma)_{\sigma \in \omega^{<\omega}}$ of parameters such that

- paths are k-**inconsistent**: $\{\varphi(x, c_{\alpha\restriction n}) : n < \omega\}$ for $\alpha \in \omega^\omega$,
- for any right-comb $X \subset \omega^{<\omega}$, $\{\varphi(x, c_\sigma) : \sigma \in X\}$ is **consistent**.

(Note the switcheroo.)

Right-combs are defined inductively:

- $\varnothing$ is a right-comb.
- X is a right-comb, every element of X extends $\sigma \frown j$, and τ extends $\sigma \frown i$ for some $i < j$, then $X \cup \{\tau\}$ is a right-comb.

Combs

A formula $\varphi(x, c)$ has the *k-comb tree property* or *k-CTP* if there is a tree $(c_\sigma)_{\sigma \in \omega^{<\omega}}$ of parameters such that

- paths are k-**inconsistent**: $\{\varphi(x, c_{\alpha \upharpoonright n}) : n < \omega\}$ for $\alpha \in \omega^\omega$,
- for any right-comb $X \subset \omega^{<\omega}$, $\{\varphi(x, c_\sigma) : \sigma \in X\}$ is **consistent**.

(Note the switcheroo.)

Right-combs are defined inductively:

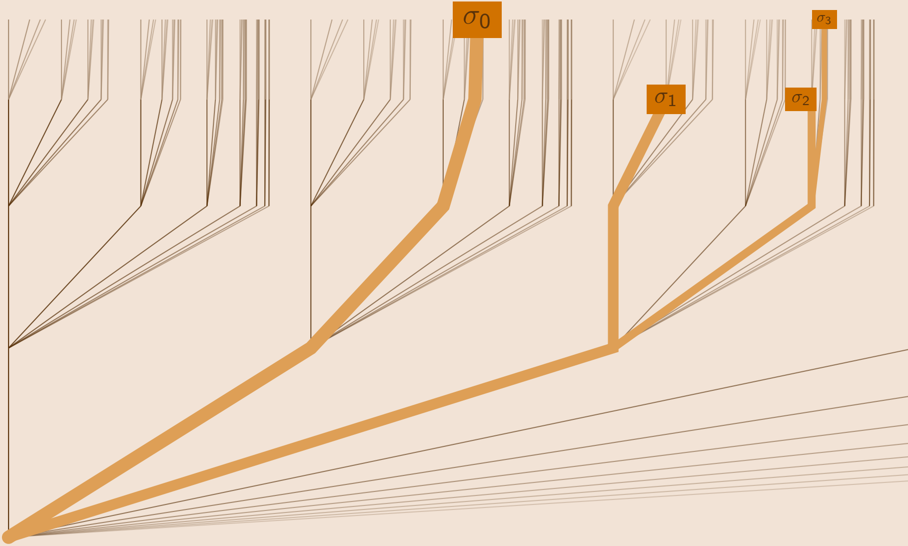
- $\varnothing$ is a right-comb.
- X is a right-comb, every element of X extends $\sigma \frown j$, and τ extends $\sigma \frown i$ for some $i < j$, then $X \cup \{\tau\}$ is a right-comb.

Mutchnik established the following in his proof that $\mathrm{NSOP}_1 = \mathrm{NSOP}_2$.

Theorem (Mutchnik)

The above condition without the switcheroo is equivalent to SOP_1.

A right-comb



Characterization of CTP

Theorem (H.)

A theory has k-CTP if and only if there is a model M, a formula $\varphi(x, b)$, and an M-heir-coheir $\mathcal{U}$ and an M-coheir $\mathcal{V}$ extending the type of b over M such that $\varphi(x, b)$ k-divides along $\mathcal{V}$ but does not divide along $\mathcal{U}$.

Characterization of CTP

Theorem (H.)

A theory has k-CTP if and only if there is a model M, a formula $\varphi(x, b)$, and an M-heir-coheir $\mathcal{U}$ and an M-coheir $\mathcal{V}$ extending the type of b over M such that $\varphi(x, b)$ k-divides along $\mathcal{V}$ but does not divide along $\mathcal{U}$.

The proof is entirely uniform in k, which leaves the following question.

Question

Does k-CTP imply 2-CTP?

Characterization of CTP

Theorem (H.)

A theory has k-CTP if and only if there is a model M, a formula $\varphi(x, b)$, and an M-heir-coheir $\mathcal{U}$ and an M-coheir $\mathcal{V}$ extending the type of b over M such that $\varphi(x, b)$ k-divides along $\mathcal{V}$ but does not divide along $\mathcal{U}$.

The proof is entirely uniform in k, which leaves the following question.

Question

Does k-CTP imply 2-CTP?

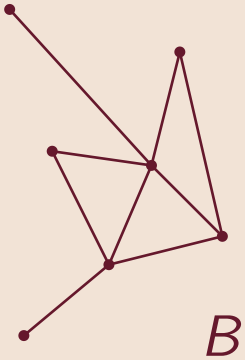
We also have the following alphabetically frustrating implication:

$$\text{ATP} \Rightarrow \text{CTP} \Rightarrow \text{BTP}$$

where the *antichain tree property* or *ATP* is another candidate for ?TP, introduced by Ahn and Kim.

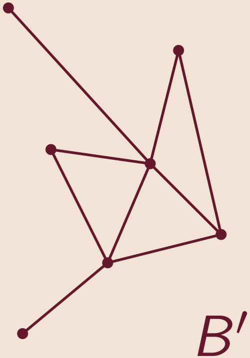
What's special about heir-coheirs?

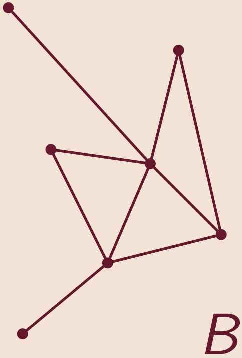
If $\mathcal{U}$ is an M-heir-coheir and B is some configuration of realizations of $\mathcal{U}$ over M, then we can find a clone B' of B with the property that every element of B' realizes $\mathcal{U}$ over $M \cup B$.



What's special about heir-coheirs?

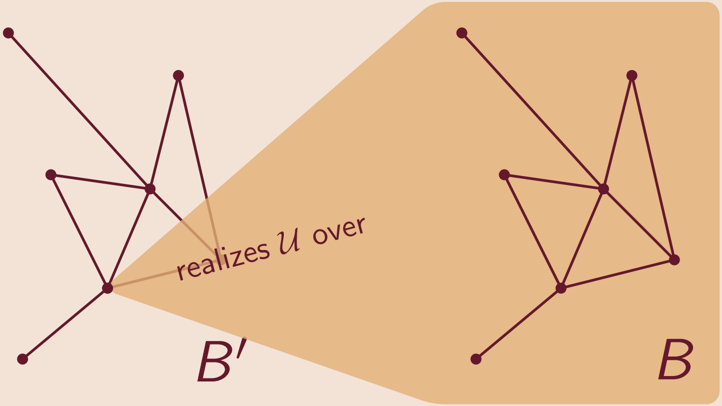
If $\mathcal{U}$ is an M-heir-coheir and B is some configuration of realizations of $\mathcal{U}$ over M, then we can find a clone B' of B with the property that every element of B' realizes $\mathcal{U}$ over $M \cup B$.





What's special about heir-coheirs?

If $\mathcal{U}$ is an M-heir-coheir and B is some configuration of realizations of $\mathcal{U}$ over M, then we can find a clone B' of B with the property that every element of B' realizes $\mathcal{U}$ over $M \cup B$.



CTP from heir-coheir $\mathcal{U}$ and coheir $\mathcal{V}$

Realize $\mathcal{U}$ •

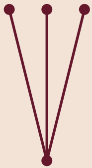
CTP from heir-coheir $\mathcal{U}$ and coheir $\mathcal{V}$

Realize $\mathcal{U}$ 

CTP from heir-coheir $\mathcal{U}$ and coheir $\mathcal{V}$

Realize $\mathcal{U}$ 

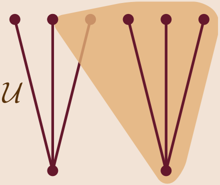
CTP from heir-coheir $\mathcal{U}$ and coheir $\mathcal{V}$



Realize $\mathcal{V}$

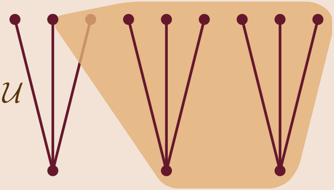
CTP from heir-coheir $\mathcal{U}$ and coheir $\mathcal{V}$

Super realize $\mathcal{U}$

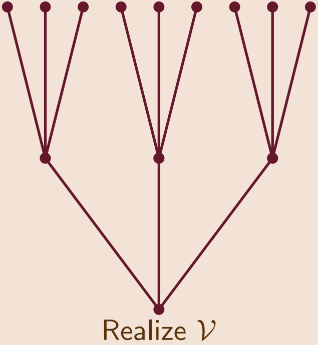


CTP from heir-coheir $\mathcal{U}$ and coheir $\mathcal{V}$

Super realize $\mathcal{U}$

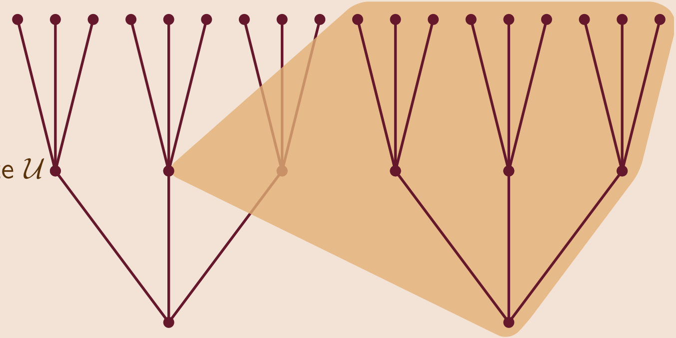


CTP from heir-coheir $\mathcal{U}$ and coheir $\mathcal{V}$

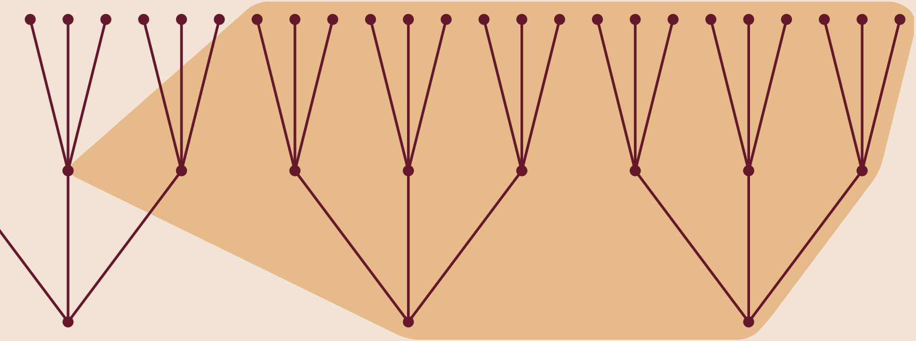


CTP from heir-coheir $\mathcal{U}$ and coheir $\mathcal{V}$

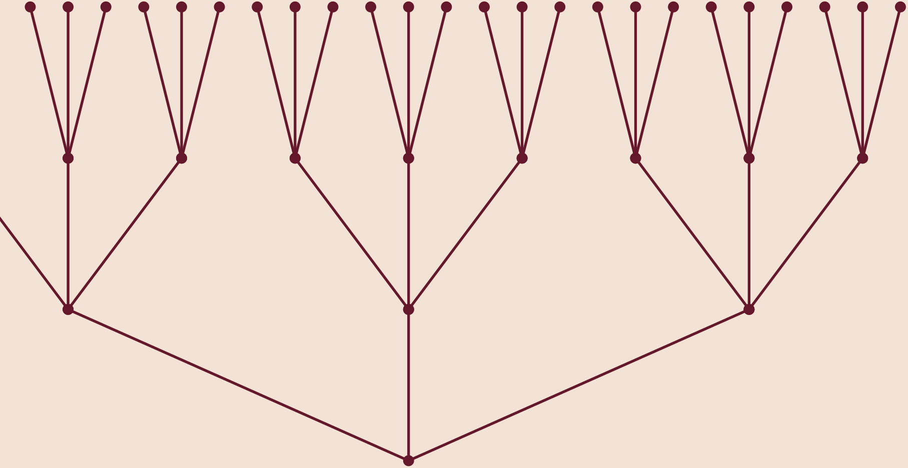
Super-duper realize $\mathcal{U}$



CTP from heir-coheir $\mathcal{U}$ and coheir $\mathcal{V}$

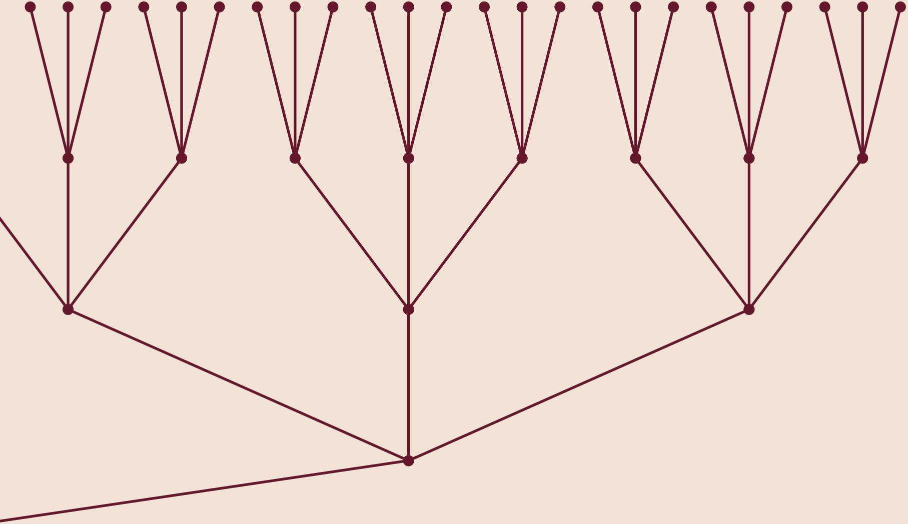


CTP from heir-coheir $\mathcal{U}$ and coheir $\mathcal{V}$

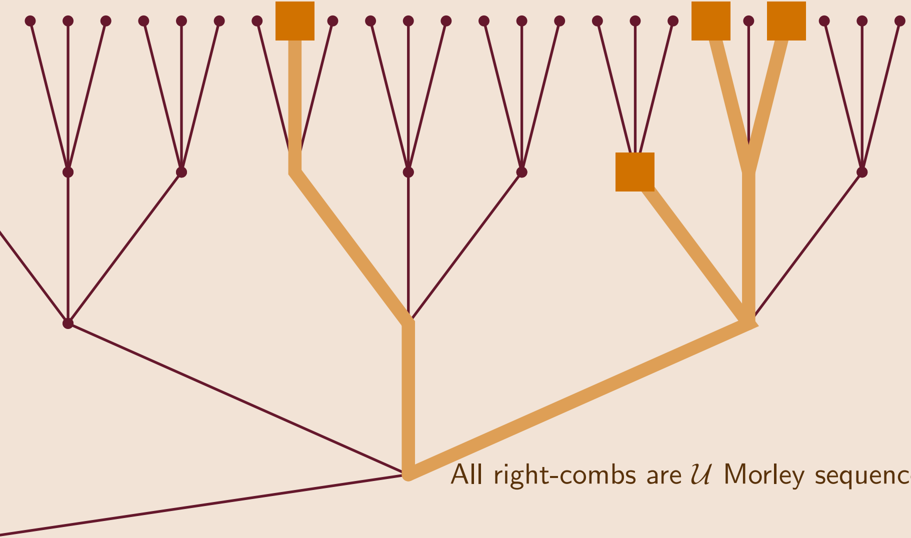


Realize $\mathcal{V}$

CTP from heir-coheir $\mathcal{U}$ and coheir $\mathcal{V}$

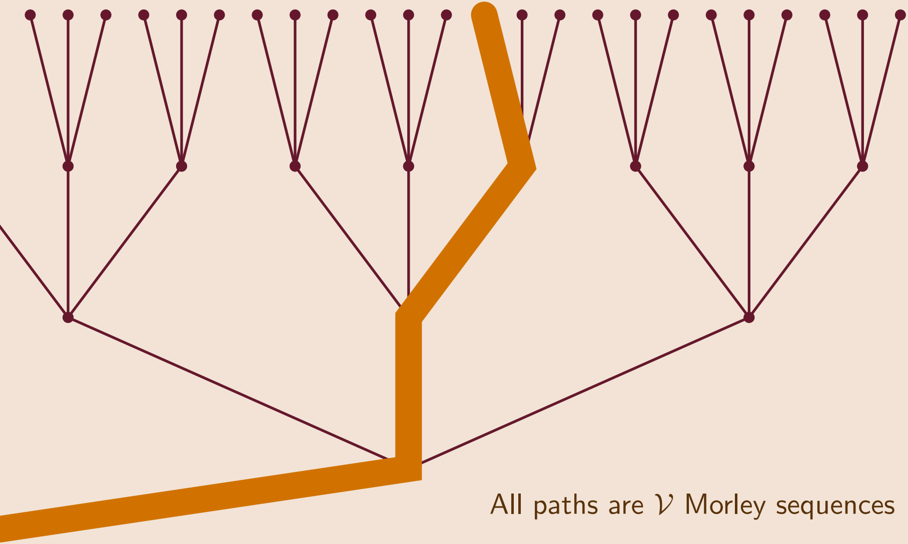


CTP from heir-coheir $\mathcal{U}$ and coheir $\mathcal{V}$



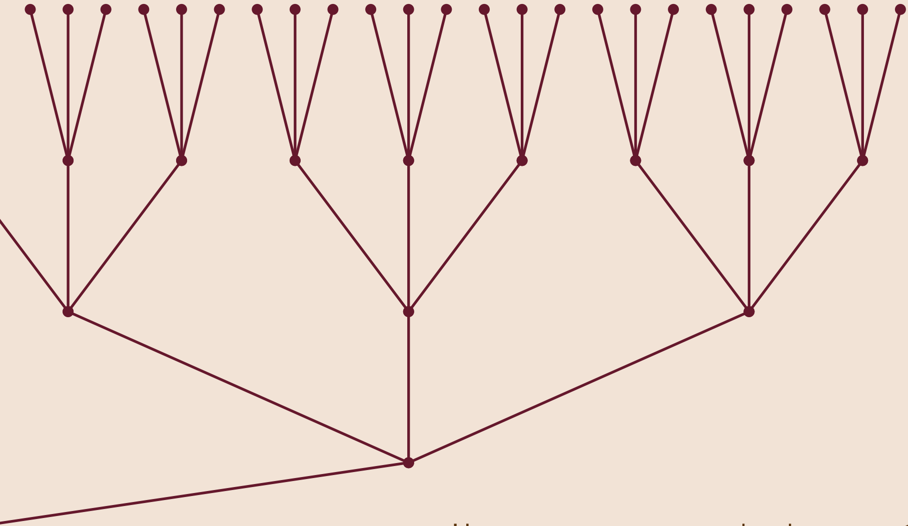
All right-combs are $\mathcal{U}$ Morley sequences

CTP from heir-coheir $\mathcal{U}$ and coheir $\mathcal{V}$



All paths are $\mathcal{V}$ Morley sequences

CTP from heir-coheir $\mathcal{U}$ and coheir $\mathcal{V}$



Use compactness to make the tree $\omega^{<\omega}$

Thank you

Forcing

Whence do heir-coheirs come?

Finding coheirs over models is trivial, but finding heir-coheirs can be hard.

Whence do heir-coheirs come?

Finding coheirs over models is trivial, but finding heir-coheirs can be hard. There are no heir-coheirs over $(\mathbb{R}, <)$ for instance.

Whence do heir-coheirs come?

Finding coheirs over models is trivial, but finding heir-coheirs can be hard. There are no heir-coheirs over $(\mathbb{R}, <)$ for instance.

The standard approach is this:

Fact

If $\mathcal{U}$ is a coheir over M and $N \succ M$ is a sufficiently saturated elementary extension, then $\mathcal{U}$ is an heir-coheir over N.

Whence do heir-coheirs come?

Finding coheirs over models is trivial, but finding heir-coheirs can be hard. There are no heir-coheirs over $(\mathbb{R}, <)$ for instance.

The standard approach is this:

Fact

If $\mathcal{U}$ is a coheir over M and $N \succ M$ is a sufficiently saturated elementary extension, then $\mathcal{U}$ is an heir-coheir over N.

This is important for the development of NTP_2 but is seemingly incompatible with the way coheirs are used in NSOP_1 (delicately building two coheirs extending the same type).

Whence do heir-coheirs come?

Finding coheirs over models is trivial, but finding heir-coheirs can be hard. There are no heir-coheirs over $(\mathbb{R}, <)$ for instance.

The standard approach is this:

Fact

If $\mathcal{U}$ is a coheir over M and $N \succ M$ is a sufficiently saturated elementary extension, then $\mathcal{U}$ is an heir-coheir over N.

This is important for the development of NTP_2 but is seemingly incompatible with the way coheirs are used in NSOP_1 (delicately building two coheirs extending the same type).

There are many heir-coheirs over $(\mathbb{Q}, <)$ (any non-realized cut). Is this generalizable?

Miniaturizing the saturation argument

Let M be a countable model of a countable theory that is a little bit saturated (computable saturation is more than enough).

Miniaturizing the saturation argument

Let M be a countable model of a countable theory that is a little bit saturated (computable saturation is more than enough).

Proposition (H.)

There is a comeager set X of non-realized types over M such that any coheir extending a type in X is an heir-coheir.

Miniaturizing the saturation argument

Let M be a countable model of a countable theory that is a little bit saturated (computable saturation is more than enough).

Proposition (H.)

There is a comeager set X of non-realized types over M such that any coheir extending a type in X is an heir-coheir.

Proof sketch.

With a finite approximation $\psi(x)$ of the type we are building generically, look to see if there is a b in the monster such that $\psi(x) \wedge \varphi(x, b)$ has infinitely many realizations in M.

Miniaturizing the saturation argument

Let M be a countable model of a countable theory that is a little bit saturated (computable saturation is more than enough).

Proposition (H.)

There is a comeager set X of non-realized types over M such that any coheir extending a type in X is an heir-coheir.

Proof sketch.

With a finite approximation $\psi(x)$ of the type we are building generically, look to see if there is a b in the monster such that $\psi(x) \wedge \varphi(x, b)$ has infinitely many realizations in M. Our little bit of saturation says that there's a $c \in M$ such that $\psi(x) \wedge \varphi(x, c)$ has infinitely many realizations in M. Commit to this as an approximation of our type.

Miniaturizing the saturation argument

Let M be a countable model of a countable theory that is a little bit saturated (computable saturation is more than enough).

Proposition (H.)

There is a comeager set X of non-realized types over M such that any coheir extending a type in X is an heir-coheir.

Proof sketch.

With a finite approximation $\psi(x)$ of the type we are building generically, look to see if there is a b in the monster such that $\psi(x) \wedge \varphi(x, b)$ has infinitely many realizations in M. Our little bit of saturation says that there's a $c \in M$ such that $\psi(x) \wedge \varphi(x, c)$ has infinitely many realizations in M. Commit to this as an approximation of our type.
Argue that if $\mathcal{U}$ extends the type we built and a realizes $\mathcal{U}$ over Mb, then every formula in the type of b over Ma is already finitely satisfiable in M by construction. $\square$

The miniaturized argument as a blueprint for CTP

That proof is a forcing argument: We have a set of conditions that we need to satisfy and we are free to satisfy them generically.

The miniaturized argument as a blueprint for CTP

That proof is a forcing argument: We have a set of conditions that we need to satisfy and we are free to satisfy them generically.
The comb tree property (even on $2^{<\omega}$ rather than $\omega^{<\omega}$) gives you precisely what you need to generically build an heir-coheir $\mathcal{U}$ that is 'shadowed' by a coheir $\mathcal{V}$ such that the given formula divides along $\mathcal{V}$ but not along $\mathcal{U}$.

The fundamental theorem of forcing

Definition

A set $X \subseteq 2^{<\omega}$ is *dense above* σ if for every τ extending σ, there is a $\mu \in X$ extending τ. X is *somewhere dense* if it is dense above some σ.

The fundamental theorem of forcing

Definition

A set $X \subseteq 2^{<\omega}$ is *dense above* σ if for every τ extending σ, there is a $\mu \in X$ extending τ. X is *somewhere dense* if it is dense above some σ.

Fact

If $X \cup Y$ is dense above σ, then either X is dense above σ or there is a τ extending σ such that Y is dense above τ.

The fundamental theorem of forcing

Definition

A set $X \subseteq 2^{<\omega}$ is *dense above* σ if for every τ extending σ, there is a $\mu \in X$ extending τ. X is *somewhere dense* if it is dense above some σ.

Fact

If $X \cup Y$ is dense above σ, then either X is dense above σ or there is a τ extending σ such that Y is dense above τ.

Proof.

Assume X is not dense above σ, then there is a τ extending σ such that X contains no elements extending τ.

The fundamental theorem of forcing

Definition

A set $X \subseteq 2^{<\omega}$ is *dense above* σ if for every τ extending σ, there is a $\mu \in X$ extending τ. X is *somewhere dense* if it is dense above some σ.

Fact

If $X \cup Y$ is dense above σ, then either X is dense above σ or there is a τ extending σ such that Y is dense above τ.

Proof.

Assume X is not dense above σ, then there is a τ extending σ such that X contains no elements extending τ. But then since $X \cup Y$ is dense above σ, it is also dense above τ, whereby Y is dense above τ. $\square$

Forcing with comb trees

Suppose we have a CTP tree $(b_\sigma)_{\sigma \in 2^{<\omega}}$ (for the formula $\varphi(x, y)$) in a mildly saturated countable model M.

Forcing with comb trees

Suppose we have a CTP tree $(b_\sigma)_{\sigma \in 2^{<\omega}}$ (for the formula $\varphi(x, y)$) in a mildly saturated countable model M. We can generically build a path $(\sigma_i)_{i<\omega}$ of elements of $2^{<\omega}$ and a filter $\mathcal{F}$ on the tree $b_{\in 2^{<\omega}}$ such that following hold:

Forcing with comb trees

Suppose we have a CTP tree $(b_\sigma)_{\sigma \in 2^{<\omega}}$ (for the formula $\varphi(x, y)$) in a mildly saturated countable model M. We can generically build a path $(\sigma_i)_{i<\omega}$ of elements of $2^{<\omega}$ and a filter $\mathcal{F}$ on the tree $b_{\in 2^{<\omega}}$ such that following hold:

- For each i, σ_{i+1} extends $\sigma_i \frown 1$.

Forcing with comb trees

Suppose we have a CTP tree $(b_\sigma)_{\sigma \in 2^{<\omega}}$ (for the formula $\varphi(x, y)$) in a mildly saturated countable model M. We can generically build a path $(\sigma_i)_{i<\omega}$ of elements of $2^{<\omega}$ and a filter $\mathcal{F}$ on the tree $b_{\in 2^{<\omega}}$ such that following hold:

- For each i, σ_{i+1} extends $\sigma_i \frown 1$.
- For each $X \in \mathcal{F}$, there is an i such that $\{b_\tau \in X : \tau \succeq \sigma_i\}$ is dense above σ_i and is in $\mathcal{F}$.

Forcing with comb trees

Suppose we have a CTP tree $(b_\sigma)_{\sigma\in 2^{<\omega}}$ (for the formula $\varphi(x,y)$) in a mildly saturated countable model M. We can generically build a path $(\sigma_i)_{i<\omega}$ of elements of $2^{<\omega}$ and a filter $\mathcal{F}$ on the tree $b_{\in 2^{<\omega}}$ such that following hold:

- For each i, σ_{i+1} extends $\sigma_i \frown 1$.
- For each $X \in \mathcal{F}$, there is an i such that $\{b_\tau \in X : \tau \succeq \sigma_i\}$ is dense above σ_i and is in $\mathcal{F}$.
- If $\psi(x,c)$ is an M-formula (with c in the monster) such that $\{b_\sigma : \psi(b_\sigma, c)\}$ has somewhere dense intersection with every element of $\mathcal{F}$, then there is a $d \in M$ such that $\{b_\sigma : \psi(b_\sigma, d)\} \in \mathcal{F}$.

Forcing with comb trees

Suppose we have a CTP tree $(b_\sigma)_{\sigma \in 2^{<\omega}}$ (for the formula $\varphi(x, y)$) in a mildly saturated countable model M. We can generically build a path $(\sigma_i)_{i<\omega}$ of elements of $2^{<\omega}$ and a filter $\mathcal{F}$ on the tree $b_{\in 2^{<\omega}}$ such that following hold:

- For each i, σ_{i+1} extends $\sigma_i \frown 1$.
- For each $X \in \mathcal{F}$, there is an i such that $\{b_\tau \in X : \tau \succeq \sigma_i\}$ is dense above σ_i and is in $\mathcal{F}$.
- If $\psi(x, c)$ is an M-formula (with c in the monster) such that $\{b_\sigma : \psi(b_\sigma, c)\}$ has somewhere dense intersection with every element of $\mathcal{F}$, then there is a $d \in M$ such that $\{b_\sigma : \psi(b_\sigma, d)\} \in \mathcal{F}$.

(Draw on chalkboard.)

Forcing with comb trees II

The second bullet point now ensures that

$$\mathcal{F} \cup \left\{ \bigcup_{i<\omega} (\text{cone above } \sigma_i \frown 0) \right\}$$

generates a non-trivial filter,

Forcing with comb trees II

The second bullet point now ensures that

$$\mathcal{F} \cup \left\{ \bigcup_{i<\omega} (\text{cone above } \sigma_i \frown 0) \right\}$$

generates a non-trivial filter, which can be extended to an ultrafilter $\mathcal{U}$ whose elements are all somewhere dense.

Forcing with comb trees II

The second bullet point now ensures that

$$\mathcal{F} \cup \left\{ \bigcup_{i<\omega} (\text{cone above } \sigma_i \frown 0) \right\}$$

generates a non-trivial filter, which can be extended to an ultrafilter $\mathcal{U}$ whose elements are all somewhere dense.

The third bullet point ensures that $\mathcal{U}$ is in fact an heir-coheir

Forcing with comb trees II

The second bullet point now ensures that

$$\mathcal{F} \cup \left\{ \bigcup_{i<\omega} (\text{cone above } \sigma_i \frown 0) \right\}$$

generates a non-trivial filter, which can be extended to an ultrafilter $\mathcal{U}$ whose elements are all somewhere dense.
The third bullet point ensures that $\mathcal{U}$ is in fact an heir-coheir and the extra set added to $\mathcal{F}$ ensures that $\varphi(x, y)$ does not divide along $\mathcal{U}$.

Forcing with comb trees II

The second bullet point now ensures that

$$\mathcal{F} \cup \left\{ \bigcup_{i<\omega} (\text{cone above } \sigma_i \frown 0) \right\}$$

generates a non-trivial filter, which can be extended to an ultrafilter $\mathcal{U}$ whose elements are all somewhere dense.
The third bullet point ensures that $\mathcal{U}$ is in fact an heir-coheir and the extra set added to $\mathcal{F}$ ensures that $\varphi(x, y)$ does not divide along $\mathcal{U}$.
Finally, let $\mathcal{V}$ be any non-principal ultrafilter on $\{b_{\sigma_i} : i < \omega\}$.

Forcing with comb trees II

The second bullet point now ensures that

$$\mathcal{F} \cup \left\{ \bigcup_{i<\omega} (\text{cone above } \sigma_i \frown 0) \right\}$$

generates a non-trivial filter, which can be extended to an ultrafilter $\mathcal{U}$ whose elements are all somewhere dense.

The third bullet point ensures that $\mathcal{U}$ is in fact an heir-coheir and the extra set added to $\mathcal{F}$ ensures that $\varphi(x, y)$ does not divide along $\mathcal{U}$.

Finally, let $\mathcal{V}$ be any non-principal ultrafilter on $\{b_{\sigma_i} : i < \omega\}$. By construction, $\varphi(x, y)$ will divide along $\mathcal{V}$.

Forcing with comb trees II

The second bullet point now ensures that

$$\mathcal{F} \cup \left\{ \bigcup_{i<\omega} (\text{cone above } \sigma_i \frown 0) \right\}$$

generates a non-trivial filter, which can be extended to an ultrafilter $\mathcal{U}$ whose elements are all somewhere dense.

The third bullet point ensures that $\mathcal{U}$ is in fact an heir-coheir and the extra set added to $\mathcal{F}$ ensures that $\varphi(x, y)$ does not divide along $\mathcal{U}$.

Finally, let $\mathcal{V}$ be any non-principal ultrafilter on $\{b_{\sigma_i} : i < \omega\}$. By construction, $\varphi(x, y)$ will divide along $\mathcal{V}$. Furthermore, the third bullet point will ensure that $\mathcal{U}$ and $\mathcal{V}$ extend the same type over M, so we have the required failure of Kim's lemma for coheirs and heir-coheirs.

Forcing with comb trees III