Forcing with model-theoretic trees

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Iowa State University

September 4, 2024
Iowa State University Logic Seminar

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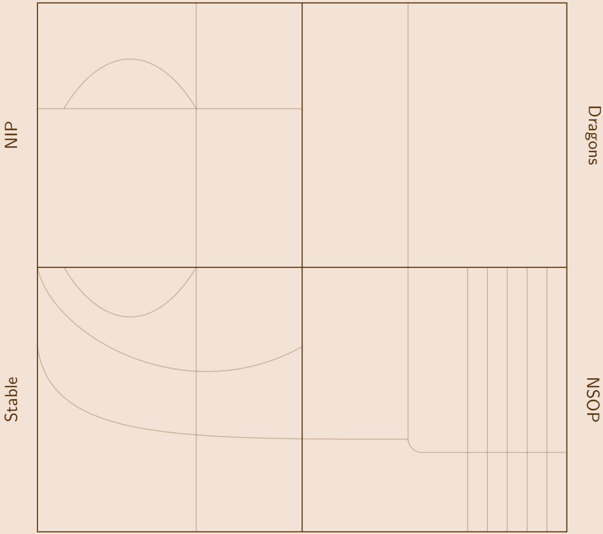
- One common endeavor in model theory is trying to find new adjectives with tractable structure theory. Example: NTP_2 is the 'least common generalization' of NIP and simplicity.

The map



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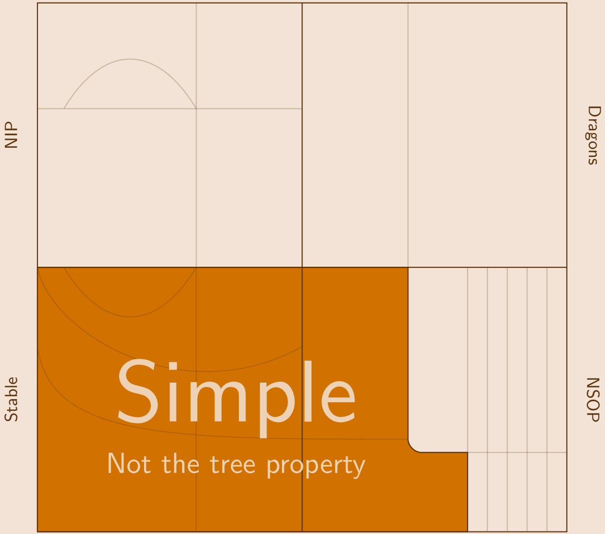

Examples:

Simple: Generic graph

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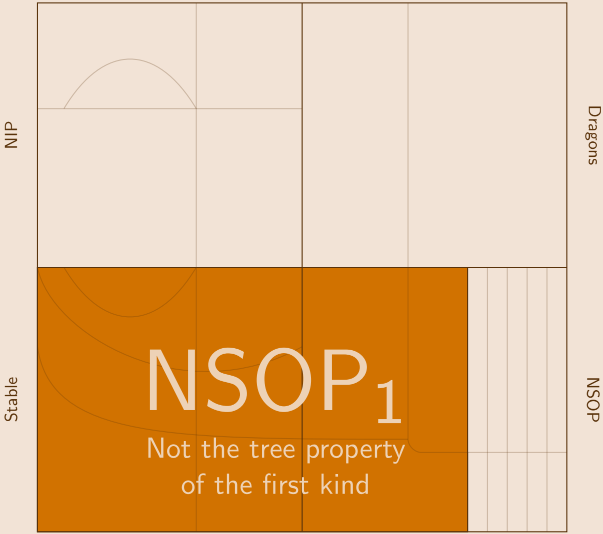

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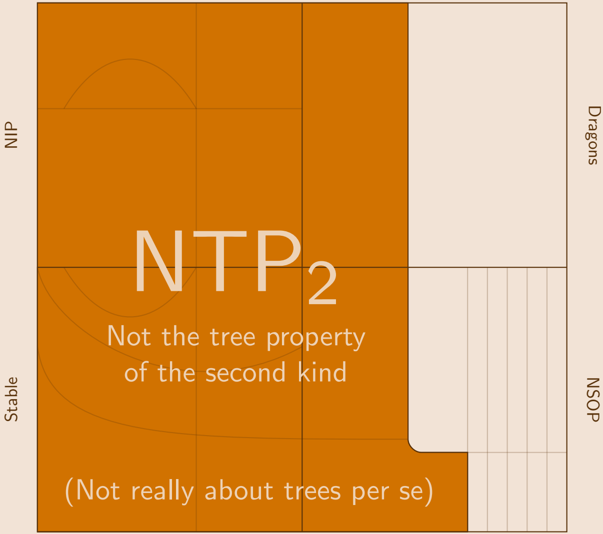

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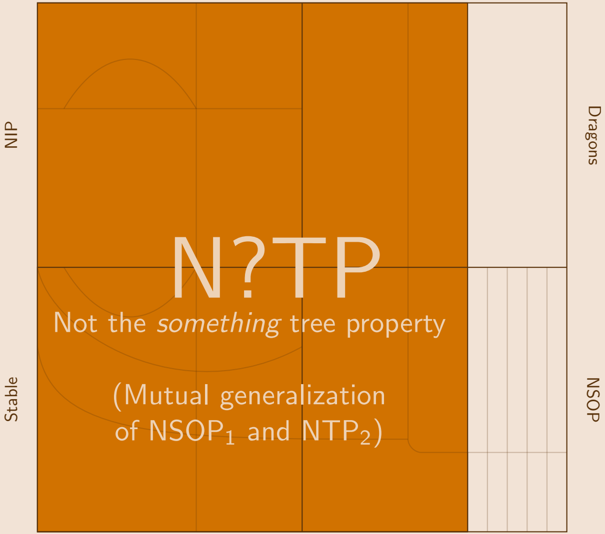

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N?TP: Generic linear order + binary function

The tree property in model theory

A formula $\varphi(x, y)$ has the *k-tree property* if there is a tree $(c_\sigma)_{\sigma \in \omega^{<\omega}}$ of parameters such that

- paths are consistent: $\{\varphi(x, c_{\alpha \restriction n}) : n < \omega\}$ for $\alpha \in \omega^\omega$,
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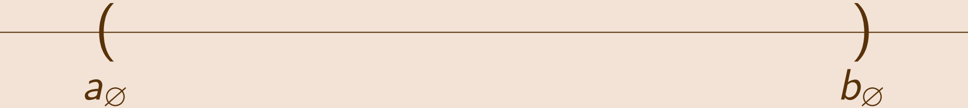

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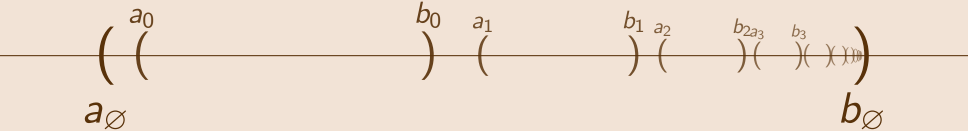

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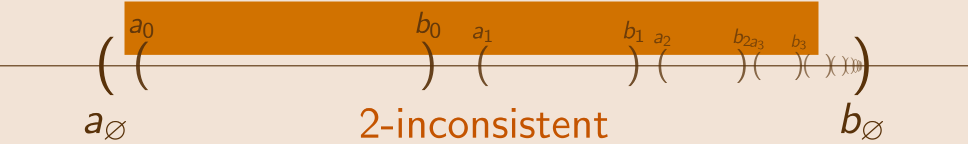

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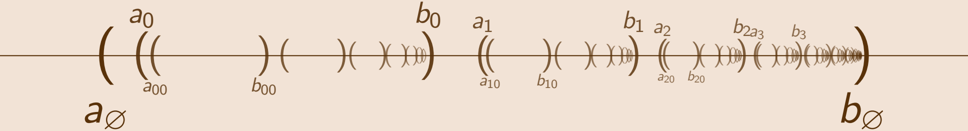
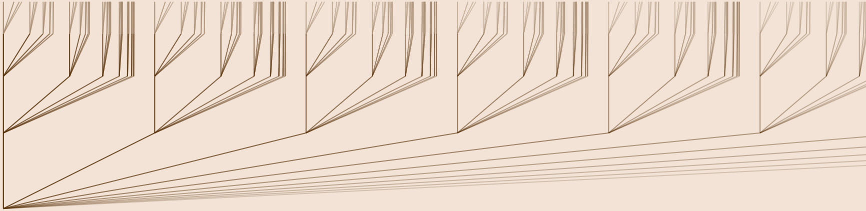


The tree in the tree property

$\omega^{<\omega}$



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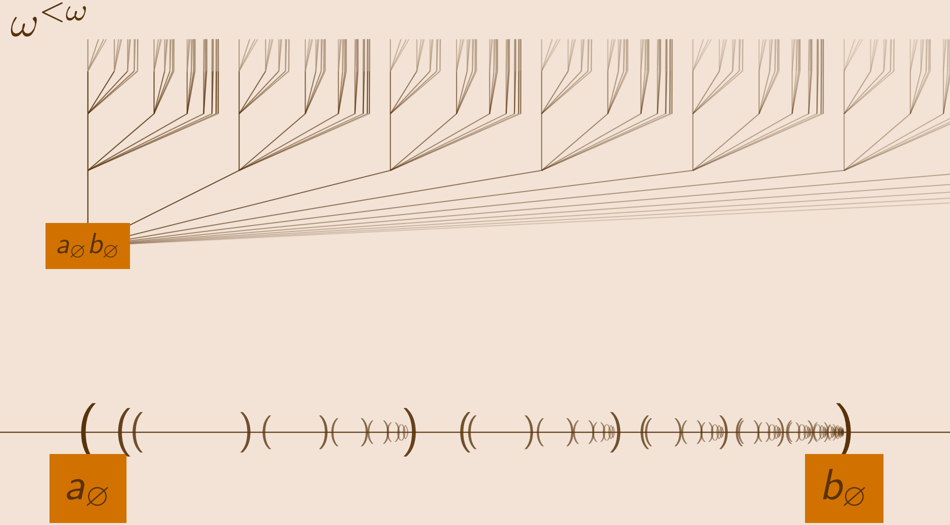

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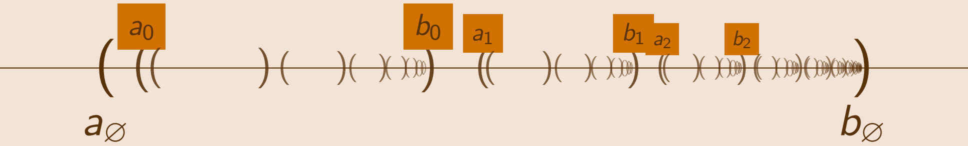

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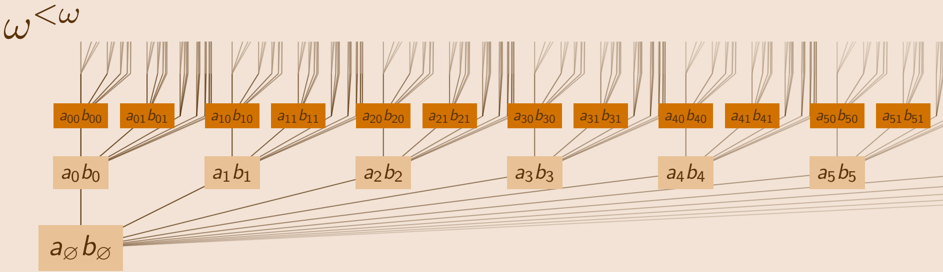

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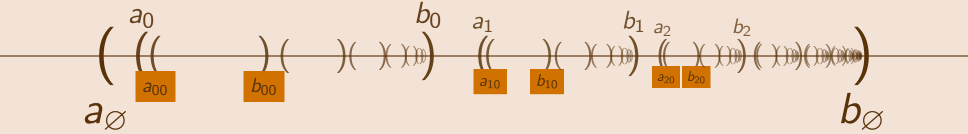

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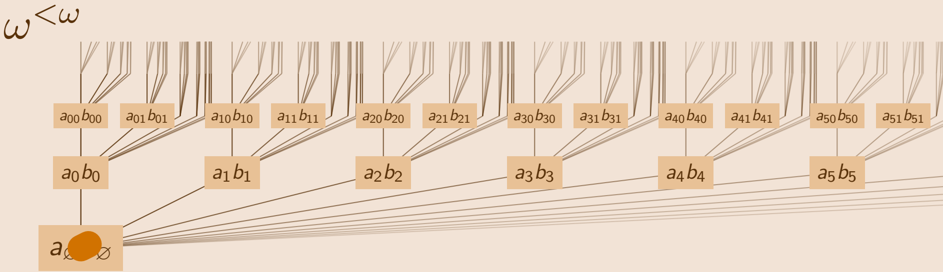

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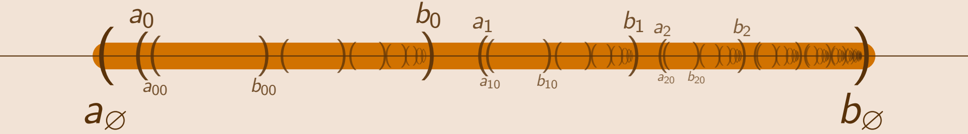

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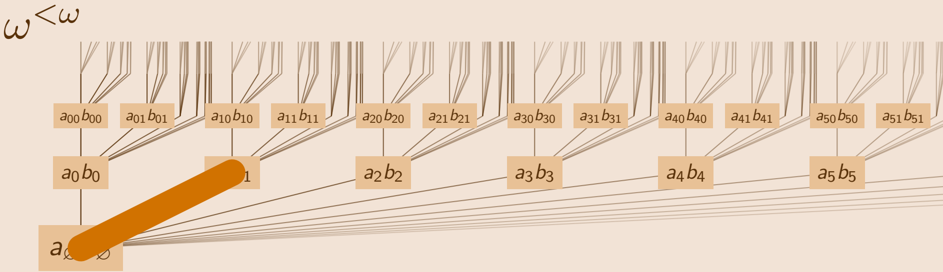

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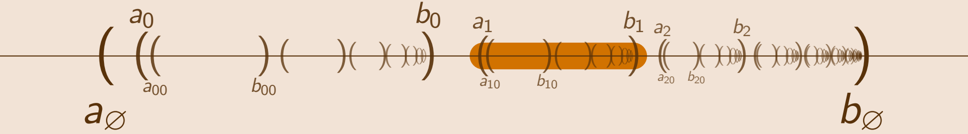

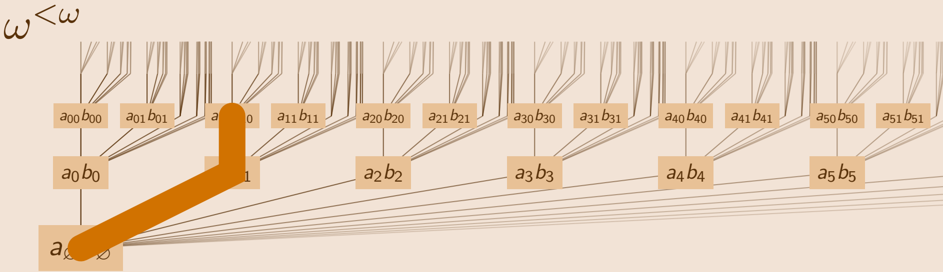

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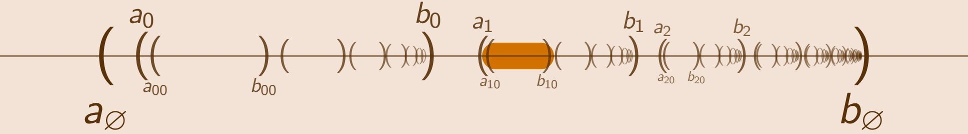

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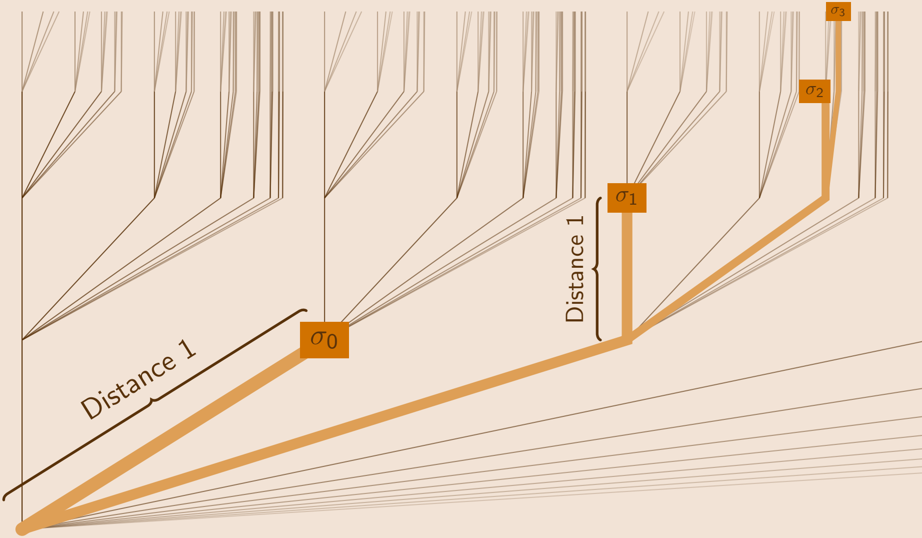
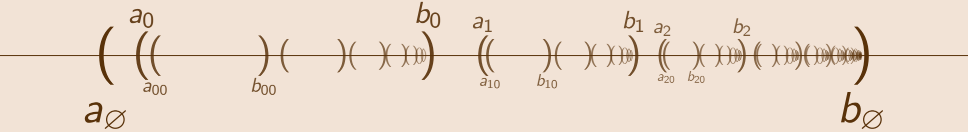
- $\varnothing$ is a short-toothed right-comb.
- X is a short-toothed right-comb, every element of X extends $\sigma \frown j$, and $i < j$, then $X \cup \{\sigma \frown i\}$ is a short-toothed right-comb.

A short-toothed right-comb



$(\mathbb{Q}, <)$ has 2-SOP$_1$

In our tree in $(\mathbb{Q}, <)$, any pair of incomparable elements are inconsistent.



Hence any short-toothed right-comb is 2-inconsistent.

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- $a_0, a_1, \ldots$ is the *Morley sequence generated by* $\mathcal{U}$.

SOP_1 in terms of coheirs

Definition

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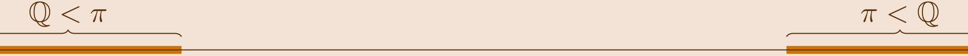
Theorem (Kaplan, Ramsey)

T has SOP_1 if and only if there is a model M, two coheirs $\mathcal{U}$ and $\mathcal{V}$ (extending the same type), and a formula $\varphi(x, y)$ such that $\varphi(x, y)$ divides along $\mathcal{U}$ but not along $\mathcal{V}$.

Coheir witnesses of SOP_1 in $(\mathbb{Q}, <)$

Two non-trivial coheirs of the 2-type living in the cut at π over $\mathbb{Q}$:



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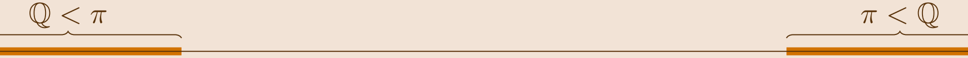

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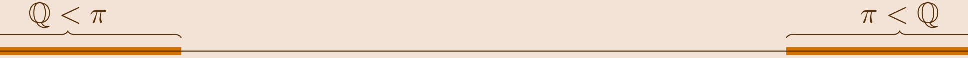

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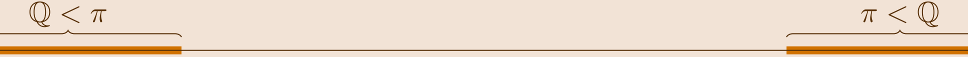

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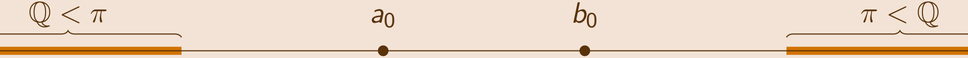

Morley sequence generated by $\mathcal{U}_{\mathsf{pinch}}$
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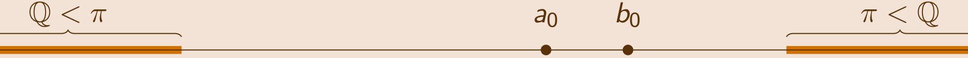

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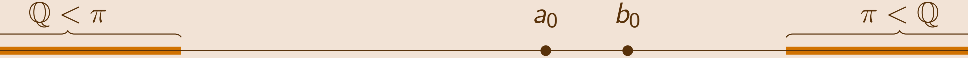

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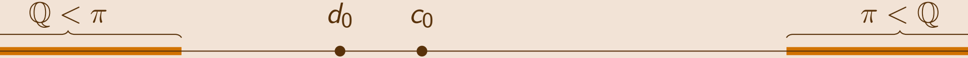

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This is non-trivial. $\mathcal{U}_{\text{pinch}}$ *does not* have this property.

TP_2 in terms of heir-coheirs

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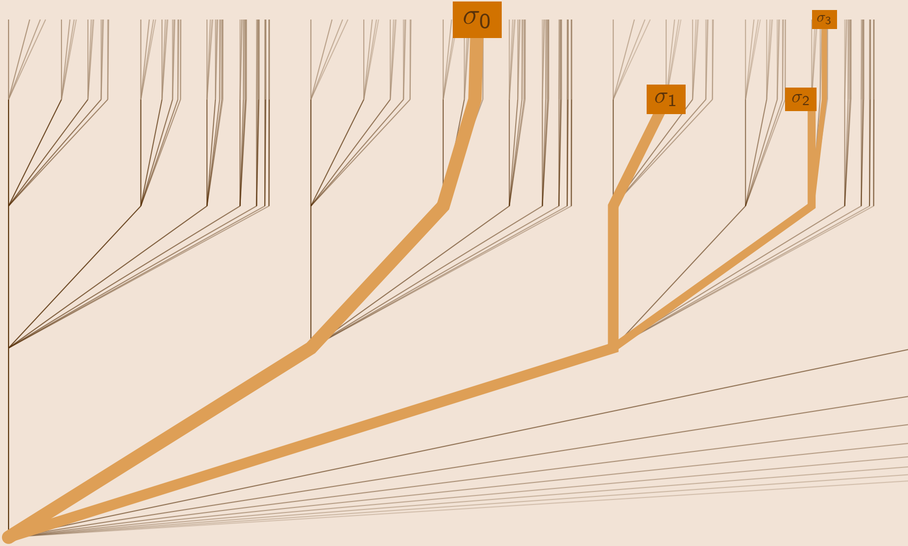
Mutchnik established the following in his proof that $\mathrm{NSOP}_1 = \mathrm{NSOP}_2$.

Theorem (Mutchnik)

The above condition without the switcheroo is equivalent to SOP_1.

A right-comb



Characterization of CTP

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A theory has k-CTP if and only if there is a model M, a formula $\varphi(x, b)$, and an M-heir-coheir $\mathcal{U}$ and an M-coheir $\mathcal{V}$ extending the type of b over M such that $\varphi(x, b)$ k-divides along $\mathcal{V}$ but does not divide along $\mathcal{U}$.

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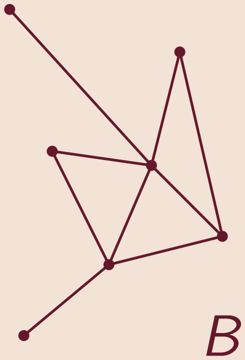
We also have the following alphabetically frustrating implication:

$$\mathrm{ATP} \Rightarrow \mathrm{CTP} \Rightarrow \mathrm{BTP}$$

where the *antichain tree property* or *ATP* is another candidate for ?TP, introduced by Ahn and Kim.

What's special about heir-coheirs?

If $\mathcal{U}$ is an M-heir-coheir and B is some configuration of realizations of $\mathcal{U}$ over M, then we can find a clone B' of B with the property that every element of B' realizes $\mathcal{U}$ over $M \cup B$.



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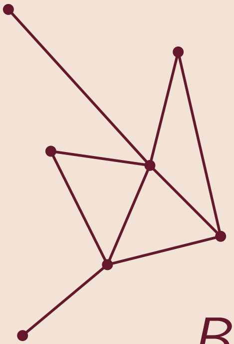

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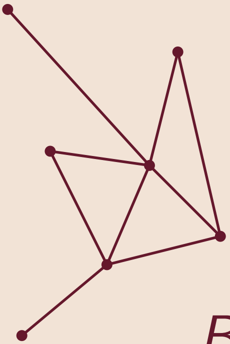

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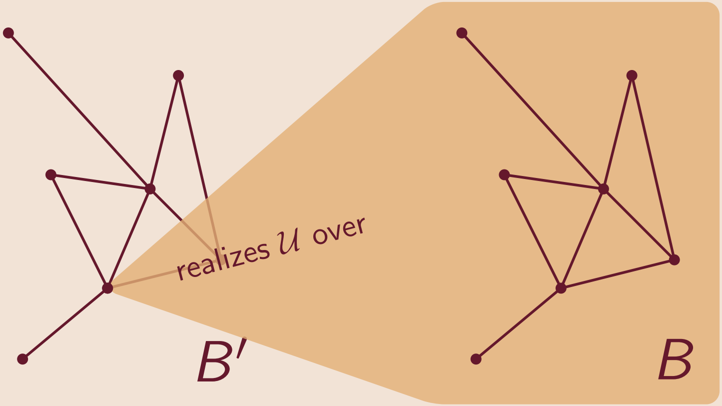

CTP from heir-coheir $\mathcal{U}$ and coheir $\mathcal{V}$

Realize $\mathcal{U}$ •

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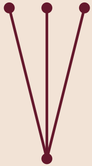

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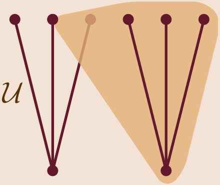

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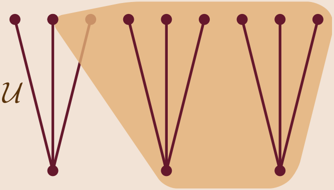

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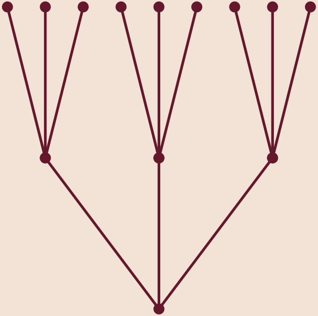

Realize $\mathcal{V}$

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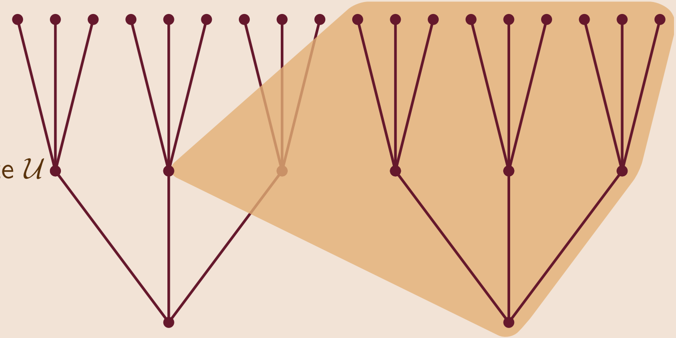

Super-duper realize $\mathcal{U}$

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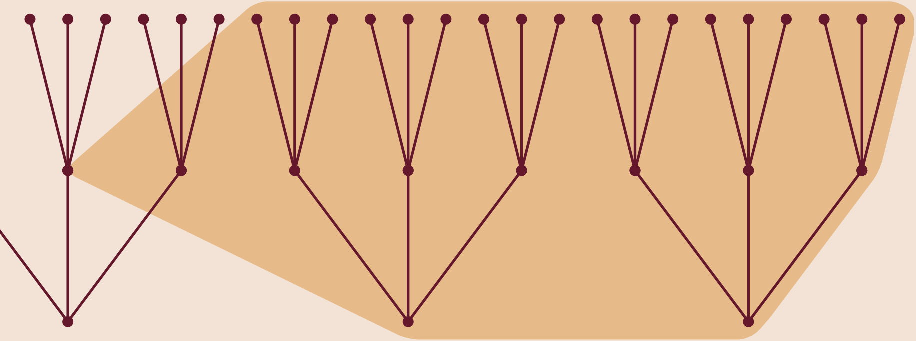

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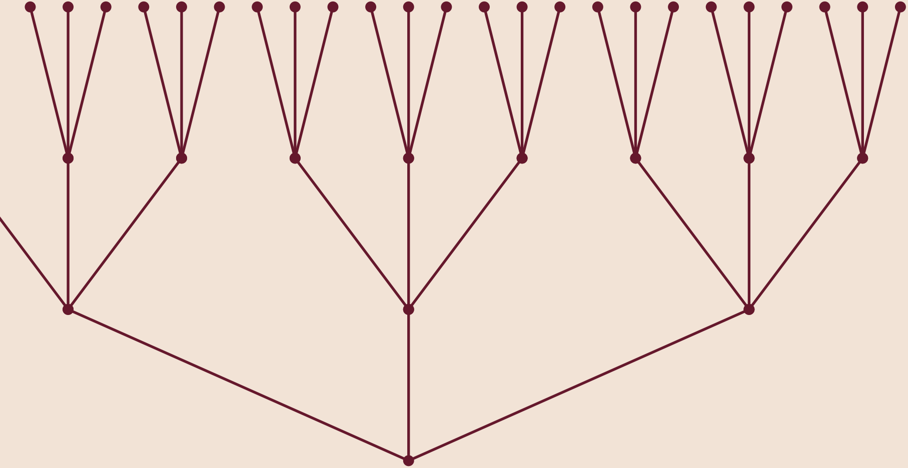

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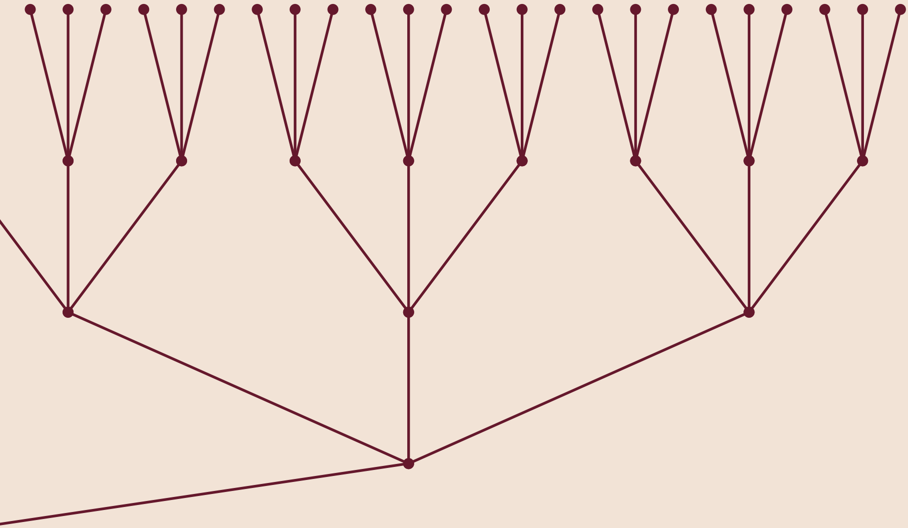

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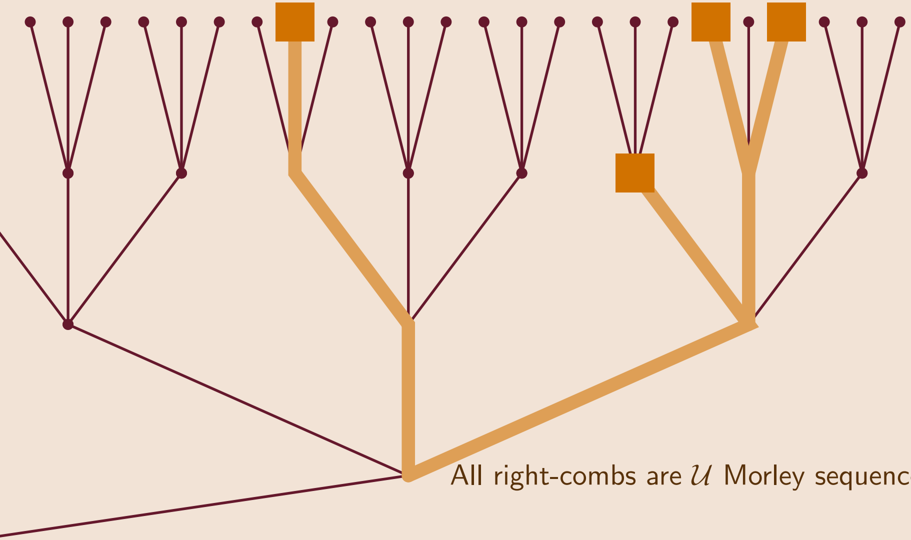

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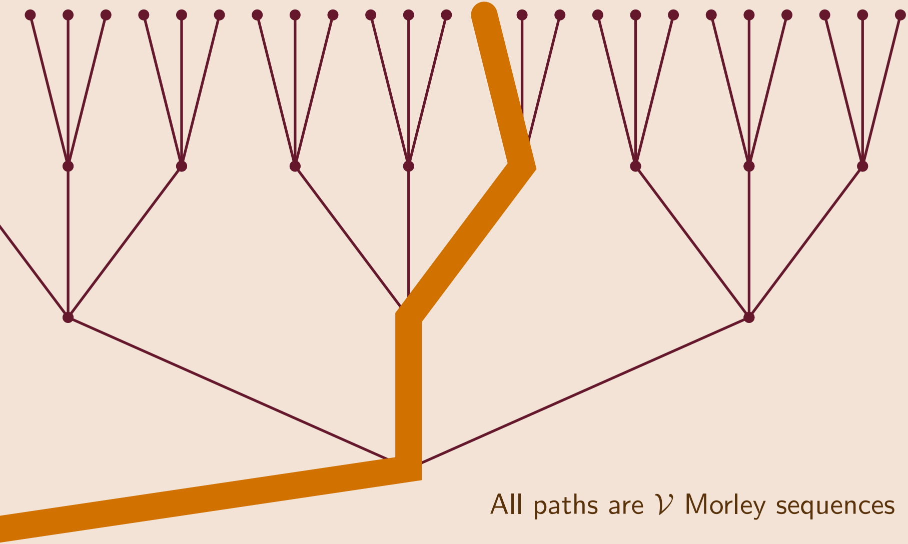

All paths are $\mathcal{V}$ Morley sequences

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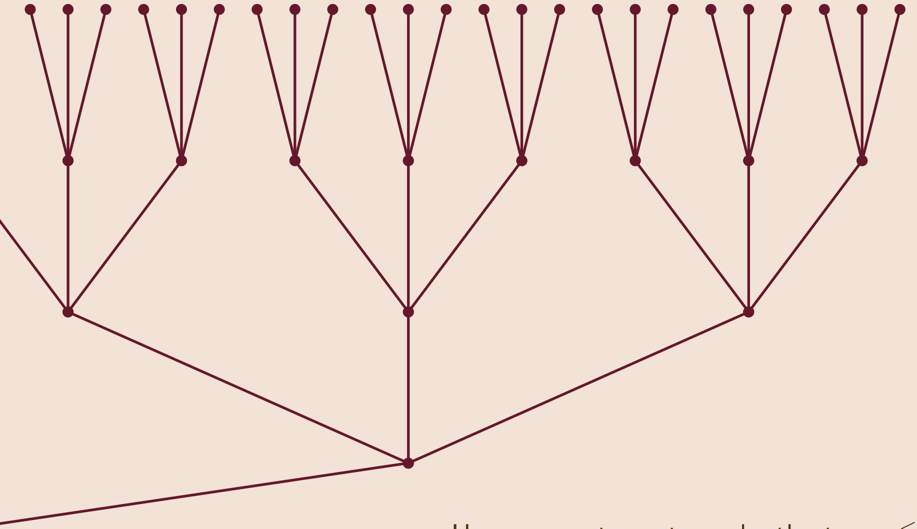

Use compactness to make the tree $\omega^{<\omega}$

Forcing

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There are many heir-coheirs over $(\mathbb{Q}, <)$ (any non-realized cut). Is this generalizable?

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Argue that if $\mathcal{U}$ extends the type we built and a realizes $\mathcal{U}$ over Mb, then every formula in the type of b over Ma is already finitely satisfiable in M by construction. $\square$

The miniaturized argument as a blueprint for CTP

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- If $\psi(x,c)$ is an M-formula (with c in the monster) such that $\{b_\sigma : \psi(b_\sigma, c)\}$ has somewhere dense intersection with every element of $\mathcal{F}$, then there is a $d \in M$ such that $\{b_\sigma : \psi(b_\sigma, d)\} \in \mathcal{F}$.

Forcing with comb trees

Suppose we have a CTP tree $(b_\sigma)_{\sigma \in 2^{<\omega}}$ (for the formula $\varphi(x, y)$) in a mildly saturated countable model M. We can generically build a path $(\sigma_i)_{i<\omega}$ of elements of $2^{<\omega}$ and a filter $\mathcal{F}$ on the tree $b_{\in 2^{<\omega}}$ such that following hold:

- For each i, σ_{i+1} extends $\sigma_i \frown 1$.
- For each $X \in \mathcal{F}$, there is an i such that $\{b_\tau \in X : \tau \succeq \sigma_i\}$ is dense above σ_i and is in $\mathcal{F}$.
- If $\psi(x, c)$ is an M-formula (with c in the monster) such that $\{b_\sigma : \psi(b_\sigma, c)\}$ has somewhere dense intersection with every element of $\mathcal{F}$, then there is a $d \in M$ such that $\{b_\sigma : \psi(b_\sigma, d)\} \in \mathcal{F}$.

(Draw on chalkboard.)

Thank you

Forcing with comb trees II

The second bullet point now ensures that

$$\mathcal{F} \cup \left\{ \bigcup_{i<\omega} (\text{cone above } \sigma_i \frown 0) \right\}$$

generates a non-trivial filter,

Forcing with comb trees II

The second bullet point now ensures that

$$\mathcal{F} \cup \left\{ \bigcup_{i<\omega} (\text{cone above } \sigma_i \frown 0) \right\}$$

generates a non-trivial filter, which can be extended to an ultrafilter $\mathcal{U}$ whose elements are all somewhere dense.

Forcing with comb trees II

The second bullet point now ensures that

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The third bullet point ensures that $\mathcal{U}$ is in fact an heir-coheir

Forcing with comb trees II

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generates a non-trivial filter, which can be extended to an ultrafilter $\mathcal{U}$ whose elements are all somewhere dense.

The third bullet point ensures that $\mathcal{U}$ is in fact an heir-coheir and the extra set added to $\mathcal{F}$ ensures that $\varphi(x, y)$ does not divide along $\mathcal{U}$.

Forcing with comb trees II

The second bullet point now ensures that

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generates a non-trivial filter, which can be extended to an ultrafilter $\mathcal{U}$ whose elements are all somewhere dense.

The third bullet point ensures that $\mathcal{U}$ is in fact an heir-coheir and the extra set added to $\mathcal{F}$ ensures that $\varphi(x, y)$ does not divide along $\mathcal{U}$.

Finally, let $\mathcal{V}$ be any non-principal ultrafilter on $\{b_{\sigma_i} : i < \omega\}$.

Forcing with comb trees II

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The third bullet point ensures that $\mathcal{U}$ is in fact an heir-coheir and the extra set added to $\mathcal{F}$ ensures that $\varphi(x, y)$ does not divide along $\mathcal{U}$.

Finally, let $\mathcal{V}$ be any non-principal ultrafilter on $\{b_{\sigma_i} : i < \omega\}$. By construction, $\varphi(x, y)$ will divide along $\mathcal{V}$.

Forcing with comb trees II

The second bullet point now ensures that

$$\mathcal{F} \cup \left\{ \bigcup_{i<\omega} (\text{cone above } \sigma_i \frown 0) \right\}$$

generates a non-trivial filter, which can be extended to an ultrafilter $\mathcal{U}$ whose elements are all somewhere dense.

The third bullet point ensures that $\mathcal{U}$ is in fact an heir-coheir and the extra set added to $\mathcal{F}$ ensures that $\varphi(x, y)$ does not divide along $\mathcal{U}$.

Finally, let $\mathcal{V}$ be any non-principal ultrafilter on $\{b_{\sigma_i} : i < \omega\}$. By construction, $\varphi(x, y)$ will divide along $\mathcal{V}$. Furthermore, the third bullet point will ensure that $\mathcal{U}$ and $\mathcal{V}$ extend the same type over M, so we have the required failure of Kim's lemma for coheirs and heir-coheirs.

Forcing with comb trees III