Bounded ultraimaginary independence

James Hanson

University of Maryland, College Park

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Something for nothing: Independence in arbitrary theories

Common themes in neo-stability

In tame contexts: Independence notion $\Rightarrow$ Generic sequences

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Q2 Can we build total $\mathop{\smile\!\!\!\!\!\!\mid}^*$-Morley sequences?

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Hyperimaginary: An equivalence class b_E of a type-definable equivalence relation E.

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With elimination of hyperimaginaries we can replace $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{b}}$ with $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{a}}$.

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- Good news: $\mathop{\smile\!\!\!\!\!\!\!\mid}^{\mathrm{bu}}$ definitely means *something*.

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- (*Walking*) For any $b' \equiv^{\mathrm{L}}_{A} b$, we have the configuration

$$b_0 \quad \equiv^{\mathrm{L}}_{Ac_1} \quad b_2 \quad \equiv^{\mathrm{L}}_{Ac_3} \quad b_4 \quad \equiv^{\mathrm{L}}_{Ac_5} \quad \cdots \quad b_{n-2} \equiv^{\mathrm{L}}_{Ac_{n-1}} \quad b_n$$

$$c_1 \quad \equiv^{\mathrm{L}}_{Ab_2} \quad c_3 \quad \equiv^{\mathrm{L}}_{Ab_4} \quad c_5 \quad \equiv^{\mathrm{L}}_{Ab_6} \quad \cdots \quad c_{n-1}$$

  where $b_0 = b$, $c_1 = c$, and $b_n = b'$.

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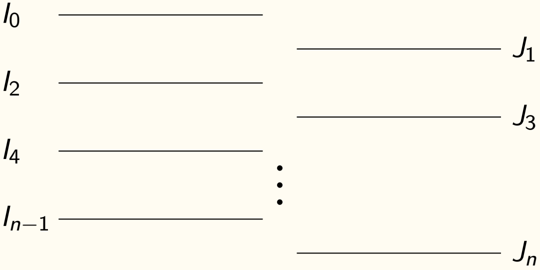
- $I \sim_A J$ if $I + J$ is A-indiscernible.
- $\approx_A$ is the equivalence relation on infinite A-indiscernible sequences generated by $\sim_A$.
- Shelah's definition in early simplicity theory: I is *based on A* if $I \equiv_A J \Leftrightarrow I \approx_A J$.

What are total $\mathop{\smile\hspace{-0.9em}\mid}^{\text{bu}}$-Morley sequences? II

Canonical witnessing configuration: $I \approx_A J$ if and only if we have



where $I_0 = I$, $J_n = J$, and $I_i + J_{i+1}$ and $I_{i+2} + J_{i+1}$ are A-indiscernible.

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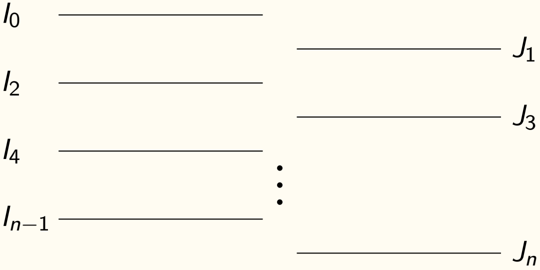

where $I_0 = I$, $J_n = J$, and $I_i + J_{i+1}$ and $I_{i+2} + J_{i+1}$ are A-indiscernible.

Theorem (H.)

$(b_i)_{i<\omega}$ is a total $\mathop{\smile\hskip-0.9em ^| \ }^{\mathrm{bu}}$-Morley sequence over A iff it is based on $\mathrm{bdd}^{\mathrm{u}}(A)$ (i.e. $I \equiv^{\mathrm{L}}_A b_{<\omega} \Leftrightarrow I \approx_A b_{<\omega}$).

Note: $I \equiv_{\mathrm{bdd}^{\mathrm{u}}(A)} J$ iff $I \equiv^{\mathrm{L}}_A J$.

The two questions

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Relationship with non-dividing

There is a 'chain condition': If $(b_i)_{i<\omega}$ is a $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{bu}}$-Morley sequence over A that is Ac-indiscernible, then $c \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{bu}}_A b_0$.

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Corollary of Corollary

In a simple theory, $(b_i)_{i<\omega}$ is a Morley sequence over A if and only if it is a total $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{bu}}$-Morley sequence over A.

In $NSOP_1$ theories

What about $NSOP_1$ theories?

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- Converse?
- Odd observation: In stable theories, you get a '$\sim_A$-distance' of 2. In simple theories, you get 3. And in NTP$_1$ theories, you get 4.

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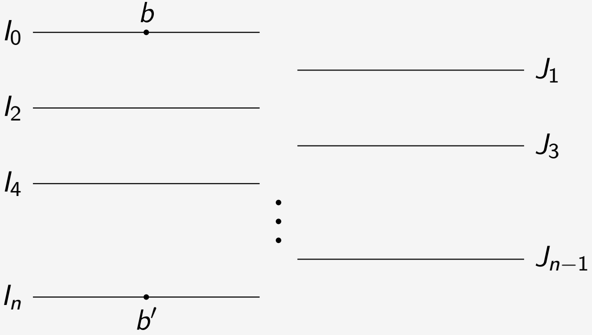
- Does this actually need large cardinals?
- Without any set theoretic hypotheses, we can get a sequence $(b_i)_{i<\omega}$ such that $b_{<i} \mathop{\smile\!\!\!\!\!\!|}^{\text{bu}}_A b_{\geq i}$ for each $i < \omega$.

Applications

Strong witnesses of Lascar strong type

Fix A and b and suppose there is a total $\mathop{\smile\kern-0.9em\raise0.4ex\hbox{|}}^{\mathrm{bu}}$-Morley sequence $I \ni b$. For any b' with $b' \equiv^{\mathrm{L}}_A b$, we have the configuration



with $I_0 = I$, $b' \in I_n$, and $I_i + J_{i+1}$ and $I_{i+2} + J_{i+1}$ A-indiscernible for all i.

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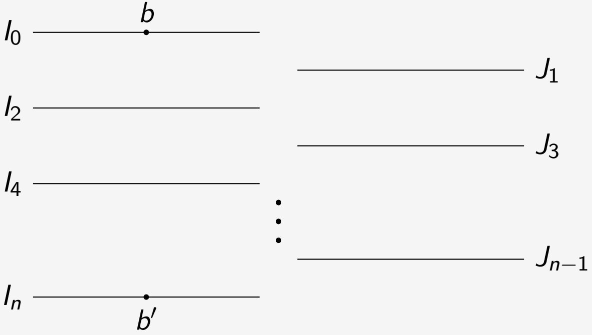

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This is similar to a configuration in the proof of the independence theorem.

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Variants of the independence theorem can generally be phrased like this:

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(T nice, maybe) Let $\Sigma(x)$ be an A-invariant partial type satisfying a chain condition. Assume that $c \models \Sigma \restriction Aab$ and $b \equiv^{\mathsf{L}}_A b'$ and that a, b, and b' are sufficiently independent of one another. Then there exists a $c' \models \Sigma \restriction Aab'$ such that $ac' \equiv_A ac$ and $b'c' \equiv_A bc$.

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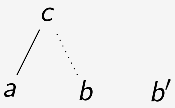

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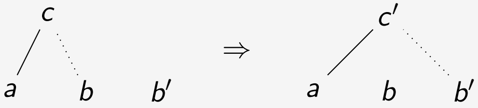

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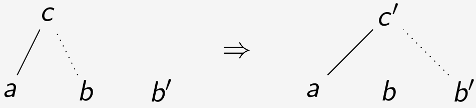

$\Sigma(x)$ is often a *generically prime* filter: If $(b_i)_{i<\omega}$ is A-indiscernible and $\Sigma(x) \vdash \varphi(x, b_0) \vee \varphi(x, b_1)$, then $\Sigma(x) \vdash \varphi(x, b_0)$.

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Can use total $\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{bu}}$-Morley sequences for notion of independence, but still need a strong chain condition, namely generic primality.

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Let $\Sigma(x)$ be A-invariant and generically prime over A. For any a, I, I', and c, if

- $I \equiv^{\mathrm{L}}_A I'$ are total $\mathop{\smile\!\!\!\!\!\!\mid}^{\text{bu}}$-Morley sequences over A,
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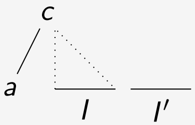

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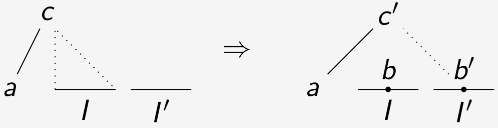

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then there are b'' and $c' \models \Sigma{\upharpoonright}Aab''$ such that $ac' \equiv_A ac$, $b''c' \equiv_A bc$, and $d^{\mathsf{L}}_A(b', b'') \leq 1$.

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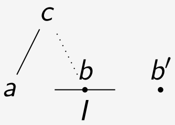

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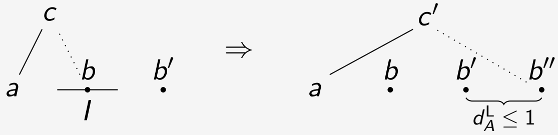

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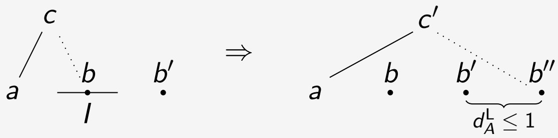

Can we weaken the generic primality requirement?

Thank you