Skolemization in Continuous Logic

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**Definition**

A discrete theory $T$ is **Skolemized** if for every formula $\varphi(\bar{x}, y)$ there is a definable function $f(\bar{x})$ such that $T \models \forall \bar{x}[\exists y \varphi(\bar{x}, y) \rightarrow \varphi(\bar{x}, f(\bar{x}))]$. 
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Consider the following terrible proof that Skolemization is possible in discrete logic: Take a theory $T$ with model $M$.
Brute Force Skolemization

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- Step I: Pass to the complete expansion, $M^\#$ (i.e., add every subset of $M^n$ for every $n < \omega$ as a predicate), and note that a complete expansion is always Skolemized.
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- **Step II:**
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$$T \models \forall \bar{x} \left[ \exists y \varphi(\bar{x}, y) \rightarrow \varphi(\bar{x}, f(\bar{x})) \right].$$

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- **Step II**: Forcing and Shoenfield absoluteness.
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- **Step I:** Pass to the *complete expansion*, $M^\#$ (i.e., add every subset of $M^n$ for every $n < \omega$ as a predicate), and note that a complete expansion is always Skolemized.

- **Step II:** Forcing and Shoenfield absoluteness.
  Argue that if a theory $T$ has an expansion $T' \supseteq T$ that is Skolemized, then there is an intermediate theory $T''$ with $T' \supseteq T'' \supseteq T$ such that $T''$ is already Skolemized and such that $|\mathcal{L}| = |\mathcal{L}''|$.
Metric signatures are defined exactly like discrete signatures with the following changes/additions:

- The symbol \( d \) instead of \( \leq \).
- To each predicate symbol \( P \) (including \( d \)), we assign a bound \( r_P > 0 \).
- To each predicate or function symbol \( s \) (other than \( d \)), we assign a modulus \( \omega_s : \mathbb{R}^+ \to \mathbb{R}^+ \), satisfying \( \omega_s(x) \to 0 \) as \( x \to 0 \).

Given a metric signature \( L \), an \( L \)-structure, \( M \), is a complete metric space \((M, d_M)\), of diameter \( \leq r_d \), with the following data:

- If \( P \) is an \( n \)-ary predicate symbol, then \( P^M : M^n \to [-r_P, r_P] \) which is \( \omega_P \)-uniformly continuous.
- If \( f \) is an \( n \)-ary function symbol, then \( f^M : M^n \to M \) which is \( \omega_f \)-uniformly continuous.
- If \( c \) is a constant symbol, then \( c^M \in M \).
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**Metric Structures**

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Terms are as in discrete logic. *Open and closed formulas*\(^1\) are defined inductively.

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Terms are as in discrete logic. *Open and closed formulas*¹ are defined inductively. Let $P\bar{t}$ and $Q\bar{s}$ be atomic formulas in the standard syntactic sense, and let $r \in \mathbb{R}$.

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- \( P\overline{t} \square r \) and \( P\overline{t} \square Q\overline{s} \) are open formulas for \( \square \in \{<, >, \neq \} \) and are closed formulas for \( \square \in \{\leq, \geq, =\} \).

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- \( P \overline{t} \sqcap r \) and \( P \overline{t} \sqcap Q \overline{s} \) are open formulas for \( \sqcap \in \{ <, >, \neq \} \) and are closed formulas for \( \sqcap \in \{ \leq, \geq, = \} \).

Let \( \varphi \)'s be open formulas and \( \chi \)'s be closed formulas.

- \( \chi \rightarrow \varphi, \neg \chi, \varphi \land \varphi', \varphi \lor \varphi', \) and \( \bigvee_{i<\omega} \varphi_i \) are open formulas.
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- \(\chi \to \varphi\), \(\neg \chi\), \(\varphi \land \varphi'\), \(\varphi \lor \varphi'\), and \(\bigvee_{i<\omega} \varphi_i\) are open formulas.
- \(\varphi \to \chi\), \(\neg \varphi\), \(\chi \land \chi'\), \(\chi \lor \chi'\), and \(\bigwedge_{i<\omega} \chi_i\) are closed formulas.
- \(\exists v \varphi\) and \(\forall v \varphi\) are open formulas.
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The semantic interpretation of any standard logical symbol is standard.

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- $M \models \forall x \varphi(\bar{a}, x)$ if for all $b \in N \geq M$, $N \models \varphi(\bar{a}, b)$. 

Examples:

- $\forall y (d_{xy} < \varepsilon \rightarrow d_{xy} = 0)$, $x$ is isolated within distance $\varepsilon$.

- $\exists xyz \forall \forall w (\chi(w) \rightarrow d_{xw} < \delta \lor d_{yw} < \delta \lor d_{zw} < \delta)$, $\chi$ can be covered by 3 open balls of radius $\delta$ (in every model).

- $\forall x \exists y (Dy = 0 \land d_{xy} = Dx)$, $D$ is the distance predicate of a set.

- $\forall x \exists y [F_{xy} = 0 \land \forall z (d_{yz} = F_{xz})]$, $F$ defines a function.
Semantics

The only non-standard symbols are strong universal quantification, $\forall x \varphi$, and weak existential quantification, $\exists x \chi$:

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$\forall i < \omega (\varphi_i(x, \bar{z}) \rightarrow \chi_i(y, \bar{z}))$, $x$ and $y$ satisfy the same formulas over $\bar{z}$, where $(\varphi_i, \chi_i)$ is a 'dense' sequence of formulas satisfying $\varphi_i(w, \bar{z}) \models \chi_i(w, \bar{z})$ (in every countable $L$).
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Given a set of closed sentences $T$ and a tuple of variables $\bar{x}$, the type space $S_{\bar{x}}(T)$ is the collection of all maximal finitely satisfiable (which, by compactness, are satisfiable) sets $p(\bar{x}) \supseteq T$ of closed formulas with free variables among $\bar{x}$.
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- For each closed formula $\chi$, let $[\chi] = \{p \in S_{\bar{x}}(T) : \chi \in p\}$.
- For each open formula $\varphi$, let $[\varphi] = \{p \in S_{\bar{x}}(T) : \neg \varphi \notin p\}$. 

The sets $[\varphi]$ form a base of a compact Hausdorff topology on $S_{\bar{x}}(T)$, even if we restrict to finitary formulas with rational bounds. For any open set $U \subseteq S_{\bar{x}}(T)$, there is an open formula $\varphi$ such that $U = [\varphi]$ iff $U$ is an $F_\sigma$ set (i.e., a $\Sigma^0_2$ set). Likewise $[\chi]$ are precisely the zerosets (i.e., the closed $G_\delta / \Pi^0_2$ sets).

If $L$ is countable, these exhaust the open and closed sets, respectively.
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Type Space

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Definition

Let $X$ be a topological spaces. An \textit{(}$X$-valued) formulas \textit{(on $\bar{x}$)} is a continuous function $f : S_{\bar{x}}(T) \rightarrow X$. 
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- $\mathbb{R}$-valued formulas are equivalent to the typical notion of formula in continuous logic.
General Formulas

Definition

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- If you squint, open and closed formulas are equivalent to $S$-valued formulas, where $S$ is the Sierpiński space.
- $\mathbb{R}$-valued formulas are equivalent to the typical notion of formula in continuous logic.
- If $F$ and $G$ are $\mathbb{R}$-valued formulas, then expressions like $F\bar{x} < r$ and $F\bar{x} + G\bar{y} = G\bar{z}$ have interpretations as open or closed formulas. We will write these freely.
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It’s far too much to ask for while preserving the metric: Let $M$ be a structure whose universe is $[0,1]$ with the standard metric and with a unary predicate $I$ such that $I^M(x) = x$. There is a formula $\phi(x,y)$ that looks like this:

$$\phi(x,0) \quad \phi(1,y) \quad \forall x \exists y \phi(x,y),$$

but there is no continuous function $f : [0,1] \to [0,1]$ such that $M \models \phi(x,f(x))$ for every $x$. 
Adding a discrete metric and Skolemizing naïvely works for some applications (Ehrenfeucht-Mostowski models).

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Weak Skolemization

If $T$ is a Skolemized theory then for any set of parameters $A$, $\text{dcl} \ A$ is a model of $T$. 
Weak Skolemization

If $T$ is a Skolemized theory then for any set of parameters $A$, $dcl A$ is a model of $T$. In discrete logic this is equivalent to being Skolemized:

**Definition**
Fix a complete theory $T$. Let $M|\models T$ and $A \subseteq M$. The definable closure of $A$, $dcl A$, is the set of all $b \in M$ such that for some $a \in A$ and some $R$-valued formula $F$, we have $M|\models \forall x (dxb = F \bar{a}x)$. 

**Definition (H.)**
A theory $T$ is weakly Skolemized if for any $A \subseteq M|\models T$, $dcl A \preceq M$. 

There are theories that are weakly Skolemized but not Skolemized.
Weak Skolemization

If $T$ is a Skolemized theory then for any set of parameters $A$, $\text{dcl} \ A$ is a model of $T$. In discrete logic this is equivalent to being Skolemized:

- For any formula $\varphi(x, y)$, for every type $p \in S_1(T)$, let $a \models p$. If $\exists y \varphi(a, y)$, then since $\text{dcl}\{a\} \models T$ there must be a $b \in \text{dcl}\{a\}$ such that $\varphi(a, b)$. Some formula $\psi(a, y)$ witnesses that $b \in \text{dcl}\{a\}$. By compactness there’s a finite list of these formulas that work for any type $p$ and we can patch these together to form a Skolem function.
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Fix a complete theory $T$. Let $M \models T$ and $A \subseteq M$. The *definable closure* of $A$, $\text{dcl} A$, is the set of all $b \in M$ such that for some $\bar{a} \in A$ and some $\mathbb{R}$-valued formula $F$, we have $M \models \forall x (dxb = F\bar{a}x)$. 
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Skolemization in Continuous Logic

November 12, 2019
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What does weak Skolemization mean?

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- By weak Skolemization, for any $\bar{a}$, if $\exists y \varphi(\bar{a}, y)$, then there is an $\mathbb{R}$-valued formula $F(\bar{x}, y)$ such that $F(\bar{a}, y)$ is the distance predicate of a singleton $\{b\}$ satisfying $\varphi(\bar{a}, b)$.
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- This is a property of $tp(\bar{a})$, but once again different types may require different formulas.
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- For each $\mathbb{R}$-valued formula $F$, the set of parameters for which it is the distance predicate of a singleton is given by the closed formula $\exists z [F\bar{x}z = 0 \land \forall y (dyz = F\bar{xy})]$. 
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- Therefore we have a covering of a compact Hausdorff space, $S_{\bar{x}}(T)$, by zero sets (i.e., closed $G_\delta / \Pi^0_2$ sets), specifically $[\neg \exists y \varphi(\bar{x}, y)]$ and the domains of definable partial Skolem functions for $\varphi$. 

When can we find a small subcover?

Question

Does there exist a $\kappa$ such that:

(*) for any compact Hausdorff space $X$ and any cover $\{F_i\}_{i \in I}$ of $X$ by closed $G_\delta$ sets there is a subcover $J \subseteq I$ such that $|J| \leq \kappa$?
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Theorem (Usuba)

A cardinal $\kappa$ has property (*) if and only if it is the first $\omega_1$-strongly compact cardinal. In particular, it is consistent that no such $\kappa$ exists.
$\omega_1$-Strongly Compact Cardinals
We don't actually need large cardinals.
We don’t actually need large cardinals.
Step II: Bringing the Cardinality Down
The Structure of Weakly Skolemized Theories
Let $F \bar{x} y$ be an $\mathbb{R}$-valued formula such that for some parameters $\bar{a}$, $F \bar{a} y$ is the distance predicate of a singleton.
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Consider the formula

$$\alpha_{F,\varepsilon}(\bar{x}) \equiv \exists y \left[ F\bar{x}y < \frac{\varepsilon}{2} \land \forall z \left( |dyz - F\bar{x}z| < \frac{\varepsilon}{2} \right) \right].$$
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$$\alpha_{F,\varepsilon}(\bar{x}) \equiv \exists y \left[ F\bar{x}y < \frac{\varepsilon}{2} \land \forall z \left( |dyz - F\bar{x}z| < \frac{\varepsilon}{2} \right) \right].$$

We have that $\alpha_{F,\varepsilon}(\bar{a})$, and while $\alpha_{F,\varepsilon}(\bar{e})$ may not guarantee that $F\bar{e}y$ is the distance predicate of a singleton, it does give that it \textit{approximately} selects out a unique element to within a distance of $\varepsilon$. 

\hspace{5cm}
Almost Functions

Let $Y$ be a set. An almost function, $f$, on $Y$ is a partial function on $X \times Y$ for some set $X$ such that for every $y \in Y$ there is an $x \in X$ such that $f(x, y)$ is defined.
Almost Functions

Let $Y$ be a set. An *almost function*, $f$, on $Y$ is a partial function on $X \times Y$ for some set $X$ such that for every $y \in Y$ there is an $x \in X$ such that $f(x, y)$ is defined.

**Definition**

An $X$-indexed continuous family of $\mathbb{R}$-valued formulas $F : X \times S_{\bar{y}z}(T) \rightarrow \mathbb{R}$ *defines a definable almost function* if for any $\bar{a}$ there is $t \in X$ such that $F_t\bar{a}z$ is the distance predicate of a singleton.
We want to show that weak Skolemization is witnessed by almost functions. We’ll need this:

**Lemma**

If $T$ is weakly Skolemized, then for any $\varepsilon > 0$ and any $\varphi(\bar{x}, y)$ and $\chi(\bar{x})$, open and closed formulas, such that $\forall \bar{x}(\chi(\bar{x}) \rightarrow \exists y \varphi(\bar{x}, y))$, there is a finite sequence of $\mathbb{R}$-valued formulas $F_0, \ldots, F_n$ and real numbers $\delta_0, \ldots, \delta_n < \varepsilon$ such that for any $\bar{a}$, if $\chi(\bar{a})$, then there is an $i \leq n$ such that $\alpha_{F_i, \delta_i}(\bar{a})$ and $\forall y(F_i \bar{a}y \leq \delta_i \rightarrow \varphi(\bar{a}, y))$. 
Building Almost Skolem Functions: The Lemma

We want to show that weak Skolemization is witnessed by almost functions. We’ll need this:

**Lemma**

If $T$ is weakly Skolemized, then for any $\varepsilon > 0$ and any $\varphi(\bar{x}, y)$ and $\chi(\bar{x})$, open and closed formulas, such that $\forall \bar{x} (\chi(\bar{x}) \rightarrow \exists y \varphi(\bar{x}, y))$, there is a finite sequence of $\mathbb{R}$-valued formulas $F_0, \ldots, F_n$ and real numbers $\delta_0, \ldots, \delta_n < \varepsilon$ such that for any $\bar{a}$, if $\chi(\bar{a})$, then there is an $i \leq n$ such that $\alpha_{F_i, \delta_i}(\bar{a})$ and $\forall y (F_i \bar{a} y \leq \delta_i \rightarrow \varphi(\bar{a}, y))$.

Recall that $\alpha_{F_i, \delta_i}(\bar{x}) \equiv \exists y \left[ F_i \bar{x} y < \frac{\delta_i}{2} \land \forall z \left( |d_{yz} - F_i \bar{x} z| < \frac{\delta_i}{2} \right) \right]$. These conditions at the end mean that $F_i \bar{a} y$ is ‘within $\delta_i$ of a distance predicate for a singleton’ and any $y$ for which $F_i \bar{a} y$ is sufficiently small is a witness to $\exists y \varphi(\bar{a}, y)$. 

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Since $T$ is weakly Skolemized, for each $p \in [\mathbf{X}]$ we can find an $\mathbb{R}$-valued formula $F_p$ such that if $\bar{a} \models p$, then $F_p\bar{a}y$ is the distance predicate of a singleton whose element witnesses $\exists y \varphi(\bar{a}, y)$. We can also find a $\delta_p > 0$ with $\delta_p < \varepsilon$, such that $\forall y (F_p\bar{a}y \leq \delta_p \rightarrow \varphi(\bar{a}, y))$, since $\varphi$ is an open formula.
Proof of Lemma

Since $T$ is weakly Skolemized, for each $p \in [\chi]$ we can find an $\mathbb{R}$-valued formula $F_p$ such that if $\bar{a} \models p$, then $F_p\bar{a}y$ is the distance predicate of a singleton whose element witnesses $\exists y \varphi(\bar{a}, y)$. We can also find a $\delta_p > 0$ with $\delta_p < \varepsilon$, such that $\forall y(F_p\bar{a}y \leq \delta_p \rightarrow \varphi(\bar{a}, y))$, since $\varphi$ is an open formula.

Now let $\beta_p(\bar{x}) = \alpha_{F_p, \delta_p}(\bar{x}) \land \forall y(F_p\bar{x}y \leq \delta_p \rightarrow \varphi(\bar{x}, y))$. Clearly by construction $p \models \beta_p$. 
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$\{[\beta_p]\}_{p \in [\chi]}$ is an open cover of $[\chi]$. By compactness it has a finite subcover indexed by $\{p_0, p_1, \ldots, p_n\}$. Now $F_i = F_{p_i}$ and $\delta_i = \delta_{p_i}$ are the required formulas and numbers.
Theorem (H.)

If $T$ is weakly Skolemized, then for any open formula $\varphi(\bar{x}, y)$ such that $T \models \forall \bar{x} \exists y \varphi(\bar{x}, y)$, there is a $2^\omega$-indexed continuous family of $\mathbb{R}$-valued formulas $F : 2^\omega \times S_{\bar{x}y}(T) \to \mathbb{R}$ that defines an almost function which produces witnesses for $\forall \bar{x} \exists y \varphi(\bar{x}, y)$ (i.e., for any $\bar{a}$ there is $t \in 2^\omega$ such that $F_t \bar{a}y$ is the distance predicate of a witness).
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Proof Idea. Use the lemma to get a finite list of approximate Skolem functions that approximately work. Build a finitely branching tree whose paths are increasingly better approximate Skolem functions. Use these to build the almost Skolem function.
Building Almost Skolem Functions: The Theorem

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Proof Idea. Use the lemma to get a finite list of approximate Skolem functions that approximately work. Build a finitely branching tree whose paths are increasingly better approximate Skolem functions. Use these to build the almost Skolem function.

Proof. We can find an open formula $\varphi'(\bar{x}, y)$ and a closed formula $\eta(\bar{x}, y)$ such that $[\varphi'] \subseteq [\eta] \subseteq [\varphi]$, $\forall \bar{x} \exists y \varphi'(\bar{x}, y)$, and $\forall \bar{x} \exists y \eta(\bar{x}, y)$. Since $[\chi]$ is a closed and a subset of $[\varphi]$, we can find an $r > 0$ such that $\forall \bar{x} \bar{y} \bar{z} w(\chi(\bar{x}, y) \land d(\bar{x} y, \bar{z} w) \leq r \to \varphi(\bar{y}, z))$. 

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Build a finitely branching subtree of $\omega^{<\omega}$: Let $\varphi_\emptyset = \varphi'$, $[\chi_\emptyset] = S_\bar{x}(T)$, and $\varepsilon_\emptyset = \frac{1}{2}r$. 
Proof of Theorem, cont.

- Build a finitely branching subtree of $\omega^{<\omega}$: Let $\varphi_\emptyset = \varphi'$, $[\chi_\emptyset] = S\bar{x}(T)$, and $\varepsilon_\emptyset = \frac{1}{2}r$.

- Given $(\varphi_\sigma, \chi_\sigma, \varepsilon_\sigma)$ for a node $\sigma$ we have by the induction hypothesis that $\forall \bar{x}(\chi_\sigma(\bar{x}) \to \exists y \varphi_\sigma(\bar{x}, y))$, so we can apply the lemma with $\varepsilon = \varepsilon_\sigma$ to get $F_{\sigma \langle 0 \rangle}, \ldots, F_{\sigma \langle n_\sigma \rangle}$ and $\delta_{\sigma \langle 0 \rangle}, \ldots, \delta_{\sigma \langle n_\sigma \rangle} < \varepsilon_\sigma$. 
Proof of Theorem, cont.

- Build a finitely branching subtree of $\omega^\omega$: Let $\varphi_\emptyset = \varphi'$, $[\chi_\emptyset] = S_{\bar{x}}(T)$, and $\varepsilon_\emptyset = \frac{1}{2}r$.

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- Let $\beta_{\sigma \downarrow i}$ be as in the proof of the lemma. (Recall: $[\beta_{\sigma \downarrow i}]$ is the set of types for which $F_{\sigma \downarrow i}$ works as an approx. Skolem function for $\varphi_{\sigma}$.)
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Let $\beta_{\sigma\downarrow i}$ be as in the proof of the lemma. (Recall: $[\beta_{\sigma\downarrow i}]$ is the set of types for which $F_{\sigma\downarrow i}$ works as an approx. Skolem function for $\varphi_\sigma$.)

The sets $[\beta_{\sigma\downarrow i}]$ cover $[\chi_\sigma]$. Let $\{\chi_{\sigma\downarrow i}\}_{i \leq n_\sigma}$ be a sequence of closed formulas such that $[\chi_{\sigma\downarrow i}] \subseteq [\beta_{\sigma\downarrow i}]$ and such that $\bigcup_{i \leq n_\sigma} [\chi_{\sigma\downarrow i}] \supseteq [\chi_\sigma]$ (such formulas always exist).
Proof of Theorem, cont.

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- Let $\varphi_{\sigma\downarrow i}(\bar{x}, y) \equiv (F_{\sigma\downarrow i}(\bar{x}, y) < \delta_{\sigma\downarrow i})$ and let $\varepsilon_{\sigma\downarrow i} = \frac{1}{2} \delta_{\sigma\downarrow i}$. 
Proof of Theorem, cont.

- Build a finitely branching subtree of $\omega^{<\omega}$: Let $\varphi_{\emptyset} = \varphi'$, $[\chi_{\emptyset}] = S_{\bar{x}}(T)$, and $\varepsilon_{\emptyset} = \frac{1}{2}r$.

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- The sets $[\beta_{\sigma \downarrow i}]$ cover $[\chi_\sigma]$. Let $\{\chi_{\sigma \downarrow i}\}_{i \leq n_\sigma}$ be a sequence of closed formulas such that $[\chi_{\sigma \downarrow i}] \subseteq [\beta_{\sigma \downarrow i}]$ and such that $\bigcup_{i \leq n_\sigma}[\chi_{\sigma \downarrow i}] \supseteq [\chi_\sigma]$ (such formulas always exist).

- Let $\varphi_{\sigma \downarrow i}(\bar{x}, y) \equiv (F_{\sigma \downarrow i}(\bar{x}, y) < \delta_{\sigma \downarrow i})$ and let $\varepsilon_{\sigma \downarrow i} = \frac{1}{2}\delta_{\sigma \downarrow i}$.

- Note that by construction we have ensured the induction hypothesis for the nodes $\sigma \downarrow i$. 
Let $R$ be the tree we built. For each path $\gamma \in [R]$ (where $[R]$ is the compact Hausdorff space of paths through $R$), let $C_\gamma = \bigcap_{n < \omega} [\chi_{\gamma|n}]$. By construction $\bigcup_{\gamma \in [R]} C_\gamma$ covers $S_{\bar{x}}(T)$. 
Let $R$ be the tree we built. For each path $\gamma \in [R]$ (where $[R]$ is the compact Hausdorff space of paths through $R$), let $C_\gamma = \bigcap_{n<\omega} [\chi_\gamma | n]$. By construction $\bigcup_{\gamma \in [R]} C_\gamma$ covers $\bar{S}_x(T)$.

Let $Q = \{ (\gamma, p) \in [R] \times S_{\bar{x}y}(T) : p \upharpoonright \bar{x} \in C_\gamma \}$. This is a closed set.
Let \( R \) be the tree we built. For each path \( \gamma \in [R] \) (where \([R]\) is the compact Hausdorff space of paths through \( R \)), let \( C_\gamma = \bigcap_{n<\omega} [x_\gamma| n] \).

By construction \( \bigcup_{\gamma \in [R]} C_\gamma \) covers \( S_{\bar{x}}(T) \).

Let \( Q = \{ (\gamma, p) \in [R] \times S_{\bar{x}y}(T) : p \restriction \bar{x} \in C_\gamma \} \). This is a closed set.

For each \( n \), let \( G^n \) be a function on \([R] \times S_{\bar{x}y}(T)\) given by \( G^n_{\gamma}(\bar{x}, y) = F_{\gamma|n}(\bar{x}, y) \).
Let $R$ be the tree we built. For each path $\gamma \in [R]$ (where $[R]$ is the compact Hausdorff space of paths through $R$), let $C_\gamma = \bigcap_{n<\omega} [\chi_{\gamma|n}]$. By construction $\bigcup_{\gamma \in [R]} C_\gamma$ covers $S_\bar{x}(T)$.

Let $Q = \{(\gamma, p) \in [R] \times S_{\bar{x}y}(T) : p \upharpoonright \bar{x} \in C_\gamma\}$. This is a closed set.

For each $n$, let $G^m_n$ be a function on $[R] \times S_{\bar{x}y}(T)$ given by $G^m_n(\bar{x}, y) = F_{\gamma|n}(\bar{x}, y)$.

If you carefully trace what we did you can show that for any $n$ we have that, for any $(\gamma, p) \in Q$, $|G^m_n(p) - G^{n+1}_{n}(p)| \leq 5 \cdot 2^{-n-2}\gamma$. So let $G : Q \to \mathbb{R}$ be the limit of the uniformly convergent sequence.
Let $R$ be the tree we built. For each path $\gamma \in [R]$ (where $[R]$ is the compact Hausdorff space of paths through $R$), let $C_\gamma = \bigcap_{n<\omega} [\chi_{\gamma|n}]$. By construction $\bigcup_{\gamma \in [R]} C_\gamma$ covers $S_{\bar{x}}(T)$.

Let $Q = \{(\gamma, p) \in [R] \times S_{\bar{x}y}(T) : p \upharpoonright \bar{x} \in C_\gamma\}$. This is a closed set.

For each $n$, let $G^n$ be a function on $[R] \times S_{\bar{x}y}(T)$ given by $G^n_\gamma(\bar{x}, y) = F_{\gamma|n}(\bar{x}, y)$.

If you carefully trace what we did you can show that for any $n$ we have that, for any $(\gamma, p) \in Q$, $|G^n_\gamma(p) - G^{n+1}_\gamma(p)| \leq 5 \cdot 2^{-n-2}r$. So let $G : Q \to \mathbb{R}$ be the limit of the uniformly convergent sequence.

Furthermore, for any $\bar{a}$ and $\gamma$ with $\bar{a} \in C_\gamma$, $G_\gamma(\bar{a}, y)$ is the distance predicate of a singleton $\{b\}$ that always has $d(b, c) \leq r$ with some $c$ such that $\varphi'(\bar{a}, c)$, and therefore $\chi(\bar{a}, c)$, holds. Hence $\varphi(\bar{a}, b)$ holds, as required.
Let $R$ be the tree we built. For each path $\gamma \in [R]$ (where $[R]$ is the compact Hausdorff space of paths through $R$), let $C_\gamma = \bigcap_{n<\omega} [x_\gamma|n]$. By construction $\bigcup_{\gamma \in [R]} C_\gamma$ covers $S_x(T)$.

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Finally, pick an embedding of $[R]$ into $2^\omega$ and use the Tietze extension theorem to continuously extend $G$ to all of $2^\omega \times S_{\bar{x}y}(T)$. □
Corollary

If $T$ has a weakly Skolemized expansion $T'$, then there is a $T''$ with $T' \supseteq T'' \supseteq T$ such that $|\mathcal{L}''| = |\mathcal{L}|$
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Proof.

If a theory has almost Skolem functions for all finitary formulas $\varphi(\bar{x}, y)$ with rational bounds that satisfy $\forall \bar{x} \exists y \varphi(\bar{x}, y)$, then it is weakly Skolemized. The number of such formulas is always at most the cardinality of the language. A definable almost function is always definable in some countable reduct. A typical iterative argument gives $T''$. \qed
Step I: Are complete expansions weakly Skolemized?
Complete Expansions

In discrete logic it is entirely trivial that complete expansions are Skolemized.
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**Definition**

If $M$ is a metric structure, the *complete expansion of $M$, $M^\#$*, is a metric structure with the same underlying domain as $M$, but with all uniformly continuous function $M^n \to \mathbb{R}$ and $M^n \to M$ added as predicates and functions.
Uniformly Locally Compact Theories

A theory $T$ is *uniformly locally compact* if for every sufficiently small $\varepsilon > 0$ and every $\delta > 0$, there is an $N(\varepsilon, \delta) < \omega$ such that every closed $\varepsilon$-ball in every model of $T$ can be covered by at most $N(\varepsilon, \delta)$ open $\delta$-balls.
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**Proposition (H.)**

If $T$ is uniformly locally compact, then any model $M \models T$ has an expansion $M'$ such that $\text{Th}(M')$ is weakly Skolemized.
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Proposition (H.)

If $T$ is uniformly locally compact, then any model $M \models T$ has an expansion $M'$ such that $\text{Th}(M')$ is weakly Skolemized.

Proof (for $\mathbb{R}$).

Suppose $T$ has a model $M$ whose underlying metric space is uniformly equivalent to $\mathbb{R}$. Add distance predicates $\{D_r\}_{r \in [0,1)}$ for each set of the form $\mathbb{Z} + r$. Since each $D_r$ is uniformly discrete, we can Skolemize it na"ively. By uniform local compactness, for every $N \models T'$ and every $a \in N$ there is an $r \in [0,1)$ such that $a \in D_r(N)$. Therefore every such $a$ is in the domain of a complete set of Skolem functions on some definable domain. It follows that $T'$ is weakly Skolemized.
Proof does not generalize I

Theorem (Milman)

Let $M$ be a metric structure based on the unit sphere of an infinite dimensional Hilbert space. There is a complete type $p \in S_1(Th(M))$ such that in some $N \succeq M$, $p(N)$ contains the unit sphere of an infinite dimensional Hilbert subspace.
Corollary (H.)

If $D$ is the distance predicate of a definable subset of $M$ whose distinct points are $(\geq \varepsilon)$-separated, then for any $a \models p$, $d(a, D) \geq \frac{\varepsilon}{2}$. 
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Proof.

Since \( p \) is a complete type, there is an \( r \) such that for any \( a \models p \), \( d(a, D) = r \). Assume that \( r < \frac{\varepsilon}{2} \) and work in a saturated enough model. Find \( b \in D \) such that for some \( a \models p \), \( d(a, b) = r \). Since \( a \) is contained in an infinite dimensional Hilbert subspace of realizations of \( p \), by Euclidean geometry there is a \( c \models p \) such that \( r < d(a, c) < \frac{\varepsilon}{2} \). There must be an \( e \in D \setminus \{a\} \) such that \( d(c, e) = r \), but this implies that \( d(a, e) \leq d(a, c) + d(c, e) < \frac{\varepsilon}{2} + r < \varepsilon \), which is a contradiction. \( \square \)
Thank you