Skolemization in Continuous Logic

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Brute Force Skolemization

Definition

A discrete theory T is *Skolemized* if for every formula $\varphi(\bar{x}, y)$ there is a definable function $f(\bar{x})$ such that $T \models \forall \bar{x}[\exists y \varphi(\bar{x}, y) \to \varphi(\bar{x}, f(\bar{x}))]$.

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- Step II: ~~Forcing and Shoenfield absoluteness~~.
  Argue that if a theory T has an expansion $T' \supseteq T$ that is Skolemized, then there is an intermediate theory T'' with $T' \supseteq T'' \supseteq T$ such that T'' is already Skolemized and such that $|\mathcal{L}| = |\mathcal{L}''|$.

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- If c is a constant symbol, then $c^M \in M$.

Formulas a different way

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Let φ's be open formulas and χ's be closed formulas.

- $\chi \to \varphi$, $\neg\chi$, $\varphi \wedge \varphi'$, $\varphi \vee \varphi'$, and $\bigvee_{i<\omega} \varphi_i$ are open formulas.
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- $\exists v \varphi$ and $\mathbb{\forall} v \varphi$ are open formulas.
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- $\exists v\varphi$ and $\forall\!\!\!/ v\varphi$ are open formulas.
- $\partial v\chi$ and $\forall v\chi$ are closed formulas.

The semantic interpretation of any standard logical symbol is standard.

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- $\bigwedge_{i<\omega}(\varphi_i(x, \bar{z}) \to \chi_i(y, \bar{z}))$, x and y satisfy the same formulas over $\bar{z}$, where (φ_i, χ_i) is a 'dense' sequence of formulas satisfying $\varphi_i(w, \bar{z}) \models \chi_i(w, \bar{z})$. ($\mathcal{L}$ countable.)

Type Space

Given a set of closed sentences T and a tuple of variables $\bar{x}$, the *type space* $S_{\bar{x}}(T)$ is the collection of all maximal finitely satisfiable (which, by compactness, are satisfiable) sets $p(\bar{x}) \supseteq T$ of closed formulas with free variables among $\bar{x}$.

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If $\mathcal{L}$ is countable, these exhaust the open and closed sets, respectively.

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- If you squint, open and closed formulas are equivalent to S-valued formulas, where S is the Sierpiński space.
- $\mathbb{R}$-valued formulas are equivalent to the typical notion of formula in continuous logic.
- If F and G are $\mathbb{R}$-valued formulas, then expressions like $F\bar{x} < r$ and $F\bar{x} + G\bar{y} = G\bar{z}$ have interpretations as open or closed formulas. We will write these freely.

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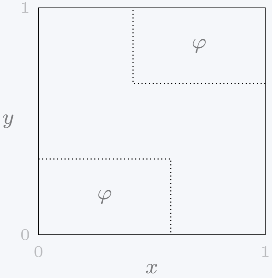

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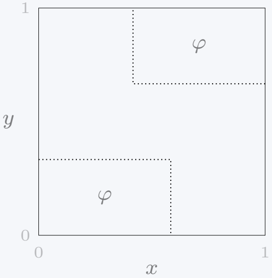

- $M \models \forall x \exists y \varphi(x, y)$, but there is no continuous function $f : [0, 1] \to [0, 1]$ such that $M \models \varphi(x, f(x))$ for every x.

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Definition

Fix a complete theory T. Let $M \models T$ and $A \subseteq M$. The *definable closure of* A, $\operatorname{dcl} A$, is the set of all $b \in M$ such that for some $\overline{a} \in A$ and some $\mathbb{R}$-valued formula F, we have $M \models \forall x(\mathrm{d}xb = F\bar{a}x)$.

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- For any formula $\varphi(x, y)$, for every type $p \in S_1(T)$, let $a \models p$. If $\exists y \varphi(a, y)$, then since $\mathrm{dcl}\{a\} \models T$ there must be a $b \in \mathrm{dcl}\{a\}$ such that $\varphi(a, b)$. Some formula $\psi(a, y)$ witnesses that $b \in \mathrm{dcl}\{a\}$. By compactness there's a finite list of these formulas that work for any type p and we can patch these togther to form a Skolem function.

Definition

Fix a complete theory T. Let $M \models T$ and $A \subseteq M$. The *definable closure of* A, $\mathrm{dcl}\, A$, is the set of all $b \in M$ such that for some $\overline{a} \in A$ and some $\mathbb{R}$-valued formula F, we have $M \models \forall x(\mathrm{d}xb = F\bar{a}x)$.

Definition (H.)

A theory T is *weakly Skolemized* if for any $A \subseteq M \models T$, $\mathrm{dcl}\, A \preceq M$.

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There are theories that are weakly Skolemized but not Skolemized.

What does weak Skolemization mean?

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- This is a property of $\mathrm{tp}(\bar{a})$, but once again different types may require different formulas.
- For each $\mathbb{R}$-valued formula F, the set of parameters for which it is the distance predicate of a singleton is given by the closed formula $\ni z[F\bar{x}z = 0 \wedge \forall y(\mathrm{d}yz = F\bar{x}y)]$.

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- Therefore we have a covering of a compact Hausdorff space, $S_{\bar{x}}(T)$, by zerosets (i.e., closed G_δ / Π^0_2 sets), specifically $[\neg\exists y \varphi(\bar{x}, y)]$ and the domains of definable partial Skolem functions for φ.

When can we find a small subcover?

Question

Does there exist a κ such that:

$(*)$ for any compact Hausdorff space X and any cover $\{F_i\}_{i \in I}$ of X by closed G_δ sets there is a subcover $J \subseteq I$ such that $|J| \leq \kappa$?

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Theorem (Usuba)

A cardinal κ has property $(*)$ if and only if it is the first ω_1-strongly compact cardinal. In particular, it is consistent that no such κ exists.

ω_1-Strongly Compact Cardinals

Overview of Large Cardinals

Overview of Large Cardinals

We don't actually need large cardinals.

Step II: Bringing the Cardinality Down
The Structure of Weakly Skolemized Theories

Approximate Functions

- Let $F\bar{x}y$ be an $\mathbb{R}$-valued formula such that for some parameters $\bar{a}$, $F\bar{a}y$ is the distance predicate of a singleton.

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- We have that $\alpha_{F,\varepsilon}(\bar{a})$, and while $\alpha_{F,\varepsilon}(\bar{e})$ may not guarantee that $F\bar{e}y$ is the distance predicate of a singleton, it does give that it *approximately* selects out a unique element to within a distance of ε.

Almost Functions

Let Y be a set. An *almost function*, f, on Y is a partial function on $X \times Y$ for some set X such that for every $y \in Y$ there is an $x \in X$ such that $f(x, y)$ is defined.

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Definition

An X-indexed continuous family of $\mathbb{R}$-valued formulas $F : X \times S_{\bar{y}z}(T) \to \mathbb{R}$ *defines a definable almost function* if for any $\bar{a}$ there is $t \in X$ such that $F_t\bar{a}z$ is the distance predicate of a singleton.

Building Almost Skolem Functions: The Lemma

We want to show that weak Skolemization is witnessed by almost functions. We'll need this:

Lemma

If T is weakly Skolemized, then for any $\varepsilon > 0$ and any $\varphi(\bar{x}, y)$ and $\chi(\bar{x})$, open and closed formulas, such that $\forall \bar{x}(\chi(\bar{x}) \to \exists y \varphi(\bar{x}, y))$, there is a finite sequence of $\mathbb{R}$-valued formulas $F_0, \ldots, F_n$ and real numbers $\delta_0, \ldots, \delta_n < \varepsilon$ such that for any $\bar{a}$, if $\chi(\bar{a})$, then there is an $i \leq n$ such that $\alpha_{F_i, \delta_i}(\bar{a})$ and $\forall y(F_i \bar{a} y \leq \delta_i \to \varphi(\bar{a}, y))$.

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Recall that $\alpha_{F_i,\delta_i}(\bar{x}) \equiv \exists y \left[F_i\bar{x}y < \frac{\delta_i}{2} \wedge \forall z \left(|\mathrm{d}yz - F_i\bar{x}z| < \frac{\delta_i}{2}\right)\right]$. These conditions at the end mean that $F_i\bar{a}y$ is 'within δ_i of a distance predicate for a singleton' and any y for which $F_i\bar{a}y$ is sufficiently small is a witness to $\exists y\varphi(\bar{a}, y)$.

Proof of Lemma

- Since T is weakly Skolemized, for each $p \in [\chi]$ we can find an $\mathbb{R}$-valued formula F_p such that if $\bar{a} \models p$, then $F_p\bar{a}y$ is the distance predicate of a singleton whose element witnesses $\exists y \varphi(\bar{a}, y)$. We can also find a $\delta_p > 0$ with $\delta_p < \varepsilon$, such that $\forall y(F_p\bar{a}y \leq \delta_p \to \varphi(\bar{a}, y))$, since φ is an open formula.

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- Now let $\beta_p(\bar{x}) = \alpha_{F_p,\delta_p}(\bar{x}) \wedge \forall y (F_p\bar{x}y \leq \delta_p \to \varphi(\bar{x}, y))$. Clearly by construction $p \models \beta_p$.

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- Now let $\beta_p(\bar{x}) = \alpha_{F_p,\delta_p}(\bar{x}) \wedge \forall y(F_p\bar{x}y \leq \delta_p \to \varphi(\bar{x}, y))$. Clearly by construction $p \models \beta_p$.
- $\{[\beta_p]\}_{p\in[\chi]}$ is an open cover of $[\chi]$. By compactness it has a finite subcover indexed by $\{p_0, p_1, \ldots, p_n\}$. Now $F_i = F_{p_i}$ and $\delta_i = \delta_{p_i}$ are the required formulas and numbers. □

Building Almost Skolem Functions: The Theorem

Theorem (H.)

If T is weakly Skolemized, then for any open formula $\varphi(\bar{x}, y)$ such that $T \models \forall\bar{x}\exists y\varphi(\bar{x}, y)$, there is a 2^ω-indexed continuous family of $\mathbb{R}$-valued formulas $F : 2^\omega \times S_{\bar{x}y}(T) \to \mathbb{R}$ that defines an almost function which produces witnesses for $\forall\bar{x}\exists y\varphi(\bar{x}, y)$ (i.e., for any $\bar{a}$ there is $t \in 2^\omega$ such that $F_t\bar{a}y$ is the distance predicate of a witness).

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Proof Idea. Use the lemma to get a finite list of approximate Skolem functions that approximately work. Build a finitely branching tree whose paths are increasingly better approximate Skolem functions. Use these to build the almost Skolem function.

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Proof Idea. Use the lemma to get a finite list of approximate Skolem functions that approximately work. Build a finitely branching tree whose paths are increasingly better approximate Skolem functions. Use these to build the almost Skolem function.

Proof. We can find an open formula $\varphi'(\bar{x}, y)$ and a closed formula $\eta(\bar{x}, y)$ such that $[\varphi'] \subseteq [\eta] \subseteq [\varphi]$, $\forall\!\!\forall \bar{x} \exists y \varphi'(\bar{x}, y)$, and $\forall \bar{x} \ni y \eta(\bar{x}, y)$.
Since $[\chi]$ is a closed and a subset of $[\varphi]$, we can find an $r > 0$ such that $\forall\!\!\forall \bar{x} y \bar{z} w (\chi(\bar{x}, y) \wedge \mathrm{d}(\bar{x} y, \bar{z} w) \leq r \to \varphi(\bar{y}, z))$.

Proof of Theorem, cont.

- Build a finitely branching subtree of $\omega^{<\omega}$: Let $\varphi_\varnothing = \varphi'$, $[\chi_\varnothing] = S_{\bar{x}}(T)$, and $\varepsilon_\varnothing = \frac{1}{2}r$.

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- Given $(\varphi_\sigma, \chi_\sigma, \varepsilon_\sigma)$ for a node σ we have by the induction hypothesis that $\forall\bar{x}(\chi_\sigma(\bar{x}) \to \exists y \varphi_\sigma(\bar{x}, y))$, so we can apply the lemma with $\varepsilon = \varepsilon_\sigma$ to get $F_{\sigma\frown 0}, \dots, F_{\sigma\frown n_\sigma}$ and $\delta_{\sigma\frown 0}, \dots, \delta_{\sigma\frown n_\sigma} < \varepsilon_\sigma$.

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- The sets $[\beta_{\sigma\frown i}]$ cover $[\chi_\sigma]$. Let $\{\chi_{\sigma\frown i}\}_{i \leq n_\sigma}$ be a sequence of closed formulas such that $[\chi_{\sigma\frown i}] \subseteq [\beta_{\sigma\frown i}]$ and such that $\bigcup_{i \leq n_\sigma} [\chi_{\sigma\frown i}] \supseteq [\chi_\sigma]$ (such formulas always exist).

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- Let $\varphi_{\sigma\frown i}(\bar{x}, y) \equiv (F_{\sigma\frown i}(\bar{x}, y) < \delta_{\sigma\frown i})$ and let $\varepsilon_{\sigma\frown i} = \frac{1}{2}\delta_{\sigma\frown i}$.

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- Note that by construction we have ensured the induction hypothesis for the nodes $\sigma \frown i$.

- Let R be the tree we built. For each path $\gamma \in [R]$ (where $[R]$ is the compact Hausdorff space of paths through R), let $C_\gamma = \bigcap_{n<\omega} [\chi_{\gamma \upharpoonright n}]$. By construction $\bigcup_{\gamma \in [R]} C_\gamma$ covers $S_{\bar{x}}(T)$.

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- If you carefully trace what we did you can show that for any n we have that, for any $(\gamma, p) \in Q$, $|G^n_\gamma(p) - G^{n+1}_\gamma(p)| \leq 5 \cdot 2^{-n-2} r$. So let $G : Q \to \mathbb{R}$ be the limit of the uniformly convergent sequence.

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- Furthermore, for any $\bar{a}$ and γ with $\bar{a} \in C_\gamma$, $G_\gamma(\bar{a}, y)$ is the distance predicate of a singleton $\{b\}$ that always has $\mathrm{d}(b, c) \leq r$ with some c such that $\varphi'(\bar{a}, c)$, and therefore $\chi(\bar{a}, c)$, holds. Hence $\varphi(\bar{a}, b)$ holds, as required.

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- Finally, pick an embedding of $[R]$ into 2^ω and use the Tietze extension theorem to continuously extend G to all of $2^\omega \times S_{\bar{x}y}(T)$. □

Step II: Bringing the Cardinality Down

Corollary

If T has a weakly Skolemized expansion T', then there is a T'' with $T' \supseteq T'' \supseteq T$ such that $|\mathcal{L}''| = |\mathcal{L}|$

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Proof.

If a theory has almost Skolem functions for all finitary formulas $\varphi(\bar{x}, y)$ with rational bounds that satisfy $\forall \bar{x} \exists y \varphi(\bar{x}, y)$, then it is weakly Skolemized. The number of such formulas is always at most the cardinality of the language. A definable almost function is always definable in some countable reduct. A typical iterative argument gives T''. □

Step I: Are complete expansions weakly Skolemized?

Complete Expansions

In discrete logic it is entirely trivial that complete expansions are Skolemized.

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Definition

If M is a metric structure, the *complete expansion of* M, $M^{\#}$, is a metric structure with the same underlying domain as M, but with all uniformly continuous function $M^n \to \mathbb{R}$ and $M^n \to M$ added as predicates and functions.

Uniformly Locally Compact Theories

A theory T is *uniformly locally compact* if for every sufficiently small $\varepsilon > 0$ and every $\delta > 0$, there is an $N(\varepsilon, \delta) < \omega$ such that every closed ε-ball in every model of T can be covered by at most $N(\varepsilon, \delta)$ open δ-balls.

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Proof (for $\mathbb{R}$).

Suppose T has a model M whose underlying metric space is uniformly equivalent to $\mathbb{R}$. Add distance predicates $\{D_r\}_{r \in [0,1)}$ for each set of the form $\mathbb{Z} + r$. Since each D_r is uniformly discrete, we can Skolemize it naïvely. By uniform local compactness, for every $N \models T'$ and every $a \in N$ there is an $r \in [0, 1)$ such that $a \in D_r(N)$. Therefore every such a is in the domain of a complete set of Skolem functions on some definable domain. It follows that T' is weakly Skolemized. □

Proof does not generalize I

Theorem (Milman)

Let M be a metric structure based on the unit sphere of an infinite dimensional Hilbert space. There is a complete type $p \in S_1(Th(M))$ such that in some $N \succeq M$, $p(N)$ contains the unit sphere of an infinite dimensional Hilbert subspace.

Proof does not generalize II

Corollary (H.)

If D is the distance predicate of a definable subset of M whose distinct points are $(\geq \varepsilon)$-separated, then for any $a \models p$, $\mathrm{d}(a, D) \geq \frac{\varepsilon}{2}$.

Proof does not generalize II

Corollary (H.)

If D is the distance predicate of a definable subset of M whose distinct points are $(\geq \varepsilon)$-separated, then for any $a \models p$, $\mathrm{d}(a, D) \geq \frac{\varepsilon}{2}$.

Proof.

Since p is a complete type, there is an r such that for any $a \models p$, $\mathrm{d}(a, D) = r$. Assume that $r < \frac{\varepsilon}{2}$ and work in a saturated enough model. Find $b \in D$ such that for some $a \models p$, $\mathrm{d}(a, b) = r$. Since a is contained in an infinite dimensional Hilbert subspace of realizations of p, by Euclidean geometry there is a $c \models p$ such that $r < \mathrm{d}(a, c) < \frac{\varepsilon}{2}$. There must be an $e \in D \setminus \{a\}$ such that $\mathrm{d}(c, e) = r$, but this implies that $\mathrm{d}(a, e) \leq \mathrm{d}(a, c) + \mathrm{d}(c, e) < \frac{\varepsilon}{2} + r < \varepsilon$, which is a contradiction. □

Thank you