Strongly Minimal Sets in Continuous Logic

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Online Logic Seminar

Continuous Logic

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- In discrete logic zerosets correspond to countably type-definable sets.
- A zeroset is *definable* if there is a formula that is the distance to it in any model (relative quantification).

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- $S_n(T)$ is compact and Hausdorff.
- $S_n(T)$ may fail to be zero-dimensional.
- Continuous function $S_n(T) \to \mathbb{R}$ correspond precisely to formulas with free variables among $\bar{x}$ (modulo T).

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- ...they do not play the same role in uncountably categorical theories:
  - Some do not have any strongly minimal sets (e.g. ∞-dim. Hilbert space).
  - Even when they do, they may only show up in imaginaries or over high dimensional models. (H.)

An Important Fact about Strongly Minimal Sets

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Strongly minimal theories behave a lot more like discrete theories than arbitrary continuous theories do.

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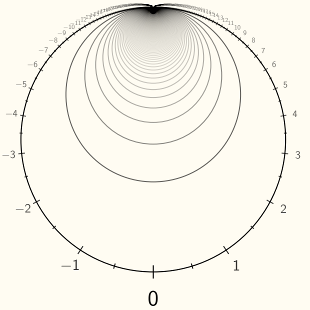
Definition

A theory T is *essentially continuous* if it does not interpret any infinite discrete structure.

An Old Strongly Minimal Set

Proposition

The theory T of $(\mathbb{R}, +)$ with the metric $\min\{|x - y|, 1\}$ is strongly minimal and essentially continuous.



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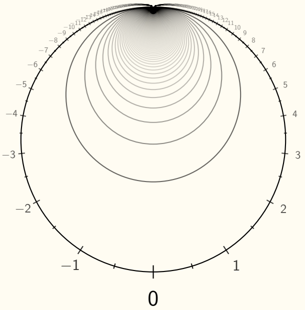

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Strong Minimality Proof.

Argue that models of T are of the form $\mathbb{R} \oplus \mathbb{Q}^\kappa$ and show that all elements realizing a non-algebraic type over some set of parameters are automorphic. □

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- By ω-stability there is a discrete strongly minimal set, D, in that sort.
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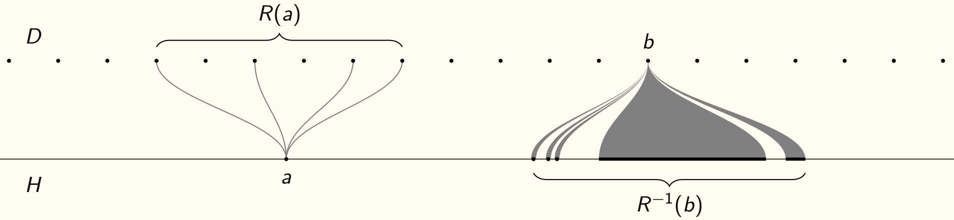
Essential Continuity Proof II



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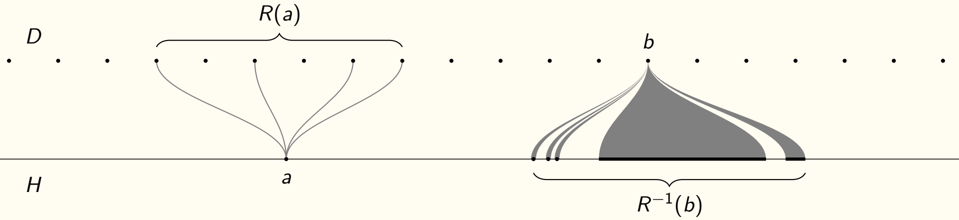

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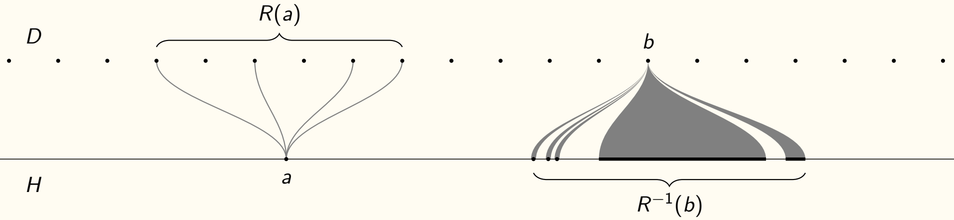

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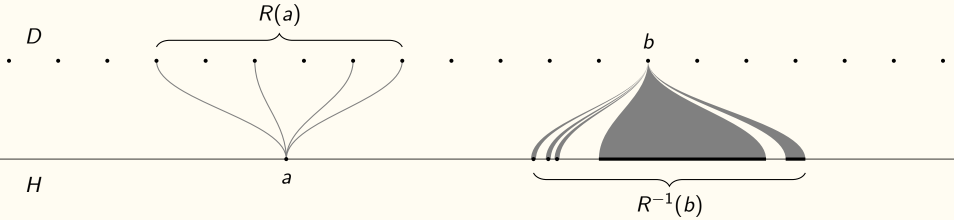

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- But, by uniform continuity of $\varphi(x, y)$, this implies that if $\varphi(a, c) = 0$, then $\varphi(a + r, c) = 0$ for all $r \in \mathbb{R}$.

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Converse?

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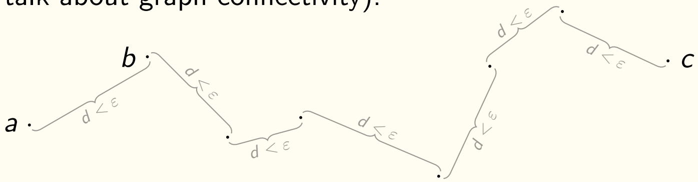

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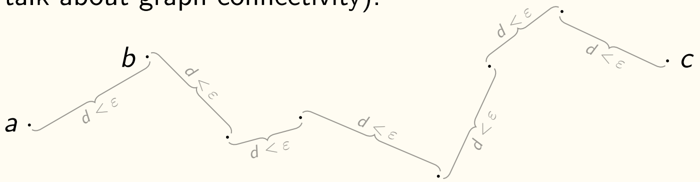

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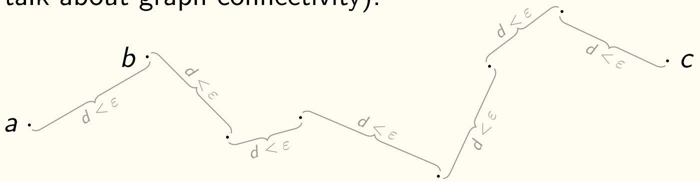

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- $x \sim_e y$ is an 'open formula' (co-zeroset).

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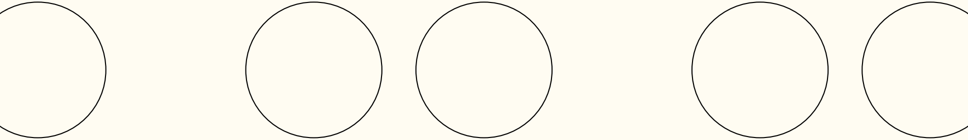

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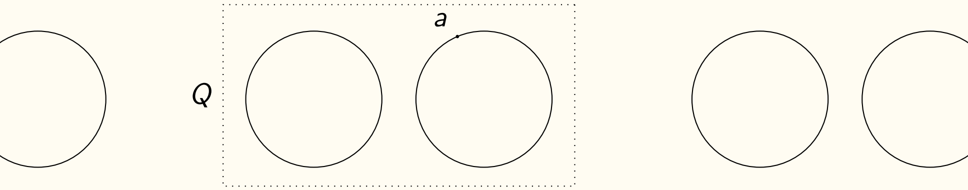

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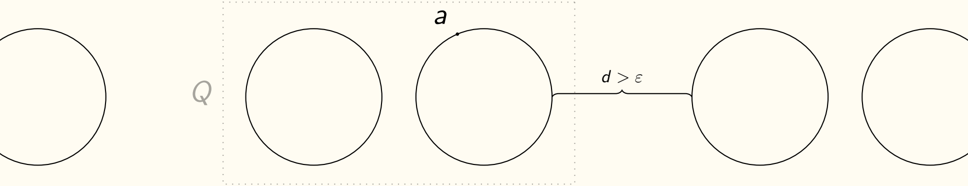

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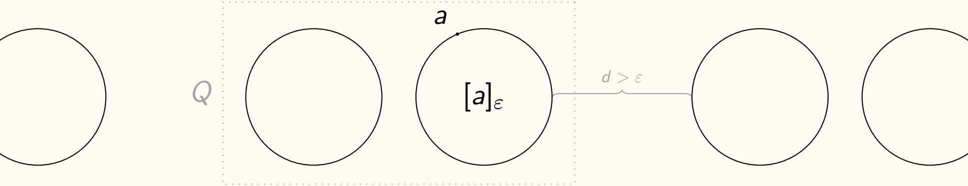

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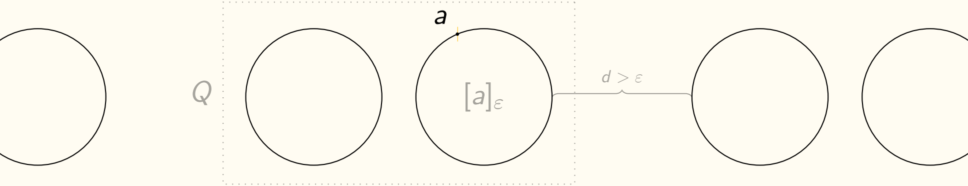
- $[a]_\varepsilon$ must be a clopen subset of Q and is therefore compact. So $\sim_\varepsilon$ is witnessed by chains of uniformly bounded length for generics.



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- if $a \not\sim_\varepsilon b$, then $\rho_n(a, b) \geq \varepsilon - \delta$.

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The generic type $p(x)$ satisfies the 'formula'

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So the formula

$$\min\left\{1, \frac{3}{2(\varepsilon - \delta)}\rho_{n+m+1}(x,y)\right\}$$

is $\{0,1\}$-valued and defines $\sim_{\frac{\varepsilon+\delta}{2}}$. The quotient is discrete and strongly minimal. □

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- Every generic element has a non-compact connected component.

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We have also shown that if the connected components of generic elements are compact then arbitrary connected components are compact.

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Let T be a strongly minimal theory. TFAE:

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Also an example showing that 'every definable set is either compact or co-pre-compact' is not good enough to be the definition of strongly minimal.

Groups

Discrete Strongly Minimal Groups

Fact (Reineke '75)

The strongly minimal groups are precisely the infinite characteristic p vector spaces and the infinite divisible Abelian groups in which for each prime p, there are finitely many elements of order p.

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where $\mathbb{Z}/p^{\infty}\mathbb{Z}$ is the p-Prüfer group (i.e. the multiplicative group of p^nth roots of unity for fixed p and arbitrary n).

Locally Compact Abelian Groups

Fact (van Kampen '35)

If G is a locally compact Abelian (Hausdorff) topological group, then it has an open subgroup H topologically isomorphic to $\mathbb{R}^n \times K$ for some compact group K and some non-negative integer n.

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- That said, G does always factor as $\mathbb{R}^n \oplus (G/\mathbb{R}^n)$.

Groups in Continuous Logic

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- There is a superstable group with a type-definable subgroup that is not the intersection of definable subgroups: $\mathrm{Th}(\mathbb{Q}, =, +, \cos, \sin)$. The subgroup $\{\cos(x) = 1, \sin(x) = 0\}$ is type-definable but not the intersection of definable subgroups. (Theory has no infinite definable proper subgroups.)

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Theorem (H.)

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In particular, G is essentially continuous if and only if it is of the second form with $n > 0$.

G can fail to be a direct product of G/K and K. An easy example is the additive group of the p-adic numbers with the metric $\min\{|x - y|_p, 1\}$.

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- If the group is divisible, then sets of order p elements must each be compact and therefore, by the classification of LCA groups, finite in G/K (where K is the compact subgroup).
- Group has non-compact connected components iff it has a non-zero power of $\mathbb{R}$. (alternatively: if it has a non-zero power of $\mathbb{R}$, then the generic and therefore everything is divisible). □

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Question

Can a Hrushovski construction or something similar build an essentially continuous strongly minimal set?

Thank you

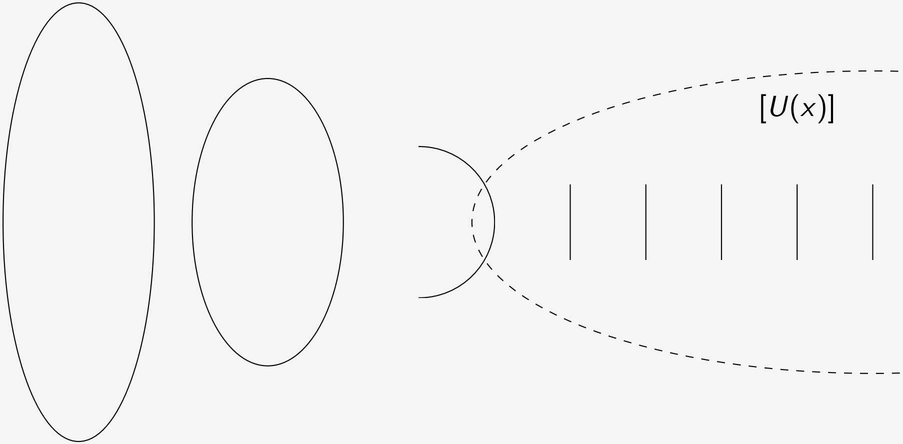
Lies I

Really the open set is:

$$U(x) \coloneqq \mathbb{\forall} y \left(\rho_n(x, y) < \frac{1}{3}(\varepsilon - \delta) \right) \vee \left(\rho_n(x, y) > \frac{2}{3}(\varepsilon - \delta) \right),$$

where $\mathbb{\forall} x$ means 'there exists x in some elementary extension.'

Lies II



Abeliantiy I

Proposition

Strongly Minimal Groups are Abelian.

- Lemma 1: Any type-definable proper subgroup in a strongly minimal group is compact.
- There must be some element g whose centralizer (subgroup of elements that commute with g), $C(g)$ is not all of the group. $C(g)$ is a zeroset, so by Lemma 1 point $C(g)$ is a compact subgroup. The orbit of g, $g^{\mathbb{Z}}$, is a subgroup of $C(g)$, so g has compact order.
- There is a natural bijection between $C(g) \setminus G$ (right coset space) and g^G (set of conjugates of g). This bijection is furthermore definable and uniformly bi-continuous.
- Not hard to show that since $C(g)$ is compact, $C(g) \setminus G$ must not be compact, implying that g^G is not compact.

Abeliantity II

- Lemma 2: If all elements of a group have 'compact order' and all non-identity elements are approximately conjugate, then the group has no more than two elements.
- For any g, h not in the centralizer, g^G and h^G are definable,* non-compact sets, so they must overlap in sufficiently saturated models. Therefore they are equal, since they are conjugacy classes. This implies that they are conjugate in G/Z (where Z is the centralizer). They also must still have compact order in G/Z so we can apply Lemma 2, and we have that G/Z is finite, implying that G is compact. Contradiction. □