Independence in arbitrary theories via automorphism groups and large cardinals

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Something for nothing:
Independence in arbitrary theories

Common themes in neo-stability

In tame contexts: Independence notion $\Rightarrow$ Generic sequences

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Q2 Can we build total $\mathop{\smile\!\!\!\!\!\!\mid}^{*}$-Morley sequences?

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- Bad news: $\mathop{\smile\!\!\!\!\!\!|}^{a}$ doesn't seem to mean much in arbitrary theories.

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Theorem (essentially Adler?)

(T simple) $(b_i)_{i \in I}$ is a Morley sequence over A iff it is a total $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{b}}$-Morley sequence over A.

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- Good news: $\mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{bu}}$ definitely means *something*.

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$(b_i)_{i<\omega}$ is a total $\mathop{\smile\!\!\!\!\!\!\,|}^{\text{bu}}$-Morley sequence over A iff it is based on $\text{bdd}^{\text{u}}(A)$ (i.e. $I \equiv^{\text{L}}_A b_{<\omega} \Leftrightarrow I \approx_A b_{<\omega}$).

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Proof.

Use the third thing I skipped for time (see slides 20-21). □

The two questions

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Horrible indiscernible tree combinatorics à la Kaplan–Ramsey. □

Corollary: Relationship with non-dividing

There is a 'chain condition': If $(b_i)_{i<\omega}$ is a $\mathop{\smile\hskip-0.9em\vert}^{\text{bu}}$-Morley sequence over A that is Ac-indiscernible, then $c \mathop{\smile\hskip-0.9em\vert}^{\text{bu}}_A b_0$.

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Corollary (H.)

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Corollary of Corollary

In a simple theory, $(b_i)_{i<\omega}$ is a Morley sequence over A if and only if it is a total $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{bu}}$-Morley sequence over A.

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- Converse?
- Odd observation: In stable theories, you get a '$\sim_A$-distance' of 2. In simple theories, you get 3. And in NSOP_1 theories, you get 4.

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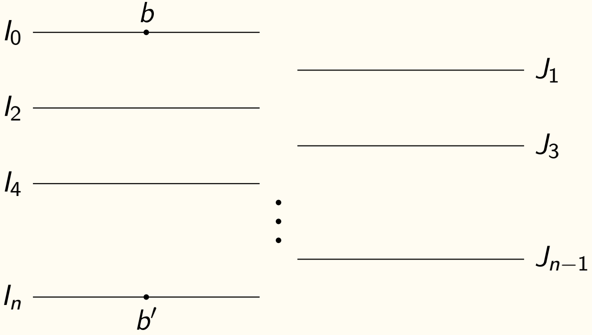
- Does this actually need large cardinals?
- Without any set theoretic hypotheses, we can get a half-infinite approximation: Sequence $(b_i)_{i<\omega}$ such that $b_{<i} \mathop{\smile\!\!\!\!\!\!\mid}^{\text{bu}}_A b_{\geq i}$ for each $i < \omega$.

Applications

Strong witnesses of Lascar strong type

Fix A and b and suppose there is a total $\mathop{\smile\hskip-0.9em\raise0.4ex\hbox{$|$}}^{\text{bu}}$-Morley sequence $I \ni b$. For any b' with $b' \equiv^{\mathrm{L}}_A b$, we have the configuration



with $I_0 = I$, $b' \in I_n$, and $I_i + J_{i+1}$ and $I_{i+2} + J_{i+1}$ A-indiscernible for all i.

Strong witnesses of Lascar strong type

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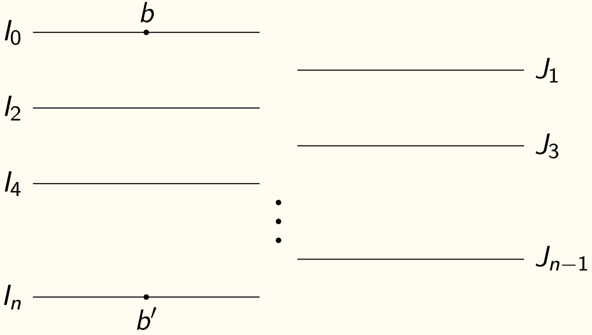

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This is similar to a configuration in the proof of the independence theorem.

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(T nice, maybe) Let $\Sigma(x)$ be an A-invariant partial type satisfying a chain condition. Assume that $c \models \Sigma \upharpoonright Aab$ and $b \equiv^{\mathsf{L}}_A b'$ and that a, b, and b' are sufficiently independent of one another. Then there exists a $c' \models \Sigma \upharpoonright Aab'$ such that $ac' \equiv_A ac$ and $b'c' \equiv_A bc$.

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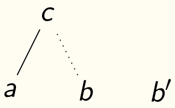

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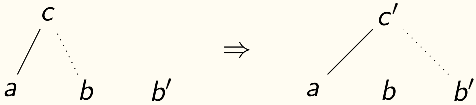

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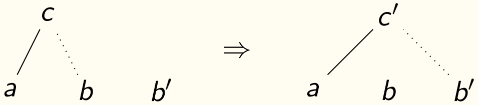

$\Sigma(x)$ is often a *generically prime* filter: If $(b_i)_{i<\omega}$ is A-indiscernible and $\Sigma(x) \vdash \varphi(x, b_0) \vee \varphi(x, b_1)$, then $\Sigma(x) \vdash \varphi(x, b_0)$.

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- $c \models \Sigma{\restriction}Aab$ for all $b \in I$, and
- $|I|, |I'| > 2^{|Aabc|+|T|}$,

then there are $b \in I$, $b' \in I'$, and $c' \models \Sigma{\restriction}Aab'$ such that $ac' \equiv_A ac$ and $b'c' \equiv_A bc$.

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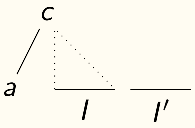

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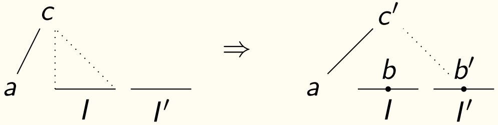

Weak amalgamation III: Off by just one (large) step

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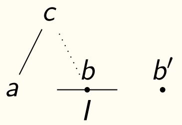

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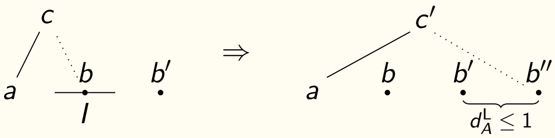

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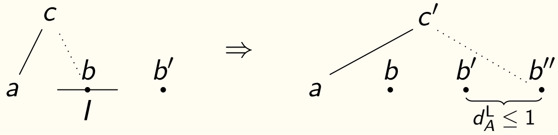

Can we weaken the generic primality requirement?

Thank you

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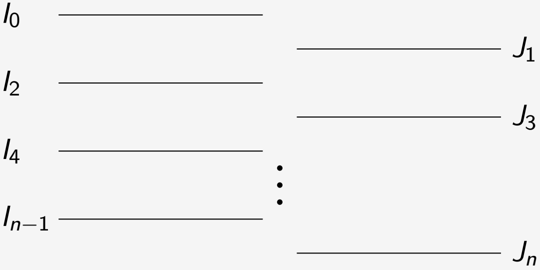
- (*Walking*) For any $b' \equiv^{\mathrm{L}}_A b$, we have the configuration

$$\begin{array}{ccccccccccccc} b_0 & \equiv^{\mathrm{L}}_{Ac_1} & b_2 & \equiv^{\mathrm{L}}_{Ac_3} & b_4 & \equiv^{\mathrm{L}}_{Ac_5} & \cdots & b_{n-2} & \equiv^{\mathrm{L}}_{Ac_{n-1}} & b_n \\ & & & & & & & & & \\ & c_1 & \equiv^{\mathrm{L}}_{Ab_2} & c_3 & \equiv^{\mathrm{L}}_{Ab_4} & c_5 & \equiv^{\mathrm{L}}_{Ab_6} & \cdots & c_{n-1} & \end{array}$$

  where $b_0 = b$, $c_1 = c$, and $b_n = b'$.

What are total $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{bu}}$-Morley sequences? II

Canonical witnessing configuration: $I \approx_A J$ if and only if we have



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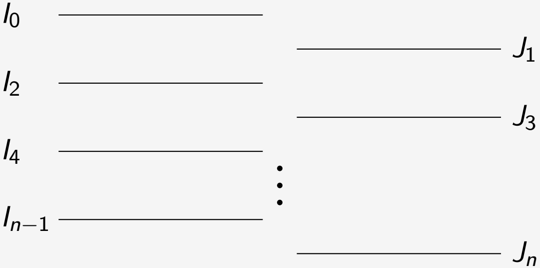

where $I_0 = I$, $J_n = J$, and $I_i + J_{i+1}$ and $I_{i+2} + J_{i+1}$ are A-indiscernible.

Theorem (H.)

$(b_i)_{i<\omega}$ is a total $\mathop{\smile\hspace{-0.9em}\mid}^{\mathrm{bu}}$-Morley sequence over A iff it is based on $\mathrm{bdd}^{\mathrm{u}}(A)$ (i.e. $I \equiv^{\mathrm{L}}_A b_{<\omega} \Leftrightarrow I \approx_A b_{<\omega}$).

Note: $I \equiv_{\mathrm{bdd}^{\mathrm{u}}(A)} J$ iff $I \equiv^{\mathrm{L}}_A J$.

Corollary: Relationship with non-dividing

There is a 'chain condition': If $(b_i)_{i<\omega}$ is a $\mathop{\smile\!\!\!\!\!\!|}^{\text{bu}}$-Morley sequence over A that is Ac-indiscernible, then $c \mathop{\smile\!\!\!\!\!\!|}^{\text{bu}}_A b_0$.

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Corollary of Corollary

In a simple theory, $(b_i)_{i<\omega}$ is a Morley sequence over A if and only if it is a total $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{bu}}$-Morley sequence over A.