Tameness and definability in continuous model theory

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Discrete model theory

A taste of model theory

Suppose we have families D_n of subsets of $\mathbb{R}^n$ satisfying:

- Each D_n is closed under Boolean combinations. (Contains $\varnothing$ and $\mathbb{R}^n$.)
- D_n's are closed under images and preimages under polynomial maps (e.g. $(x, y) \mapsto (x + y, x - 2y, xy)$).

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- If $\{(x, e^x) : x \in \mathbb{R}\} \in D_2$, then it can be that everything is a finite union of connected smooth manifolds.
- If $\{(x, \cos(x)) : x \in \mathbb{R}\} \in D_2$, then we have every Borel subset of each $\mathbb{R}^n$.

Discrete first-order logic

- A *language* is a collection of constant, function, and relation symbols.
- Example: 0,1, $+$, $\cdot$, $<$ (language of ordered semirings)
- A *structure* is a set with interpretations of the symbols.
- Example: $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$.
- Different structures can *satisfy* different *sentences* in first-order logic:

	$\mathbb{N}$	$\mathbb{Z}$	$\mathbb{Q}$	$\mathbb{R}$	
$\forall x \forall y (x < y \vee x = y \vee y < x)$	✓	✓	✓	✓	'$<$ is linear'
$\exists x (x + 1 = 0)$		✓	✓	✓	'-1 exists'
$\forall x (x \neq 0 \rightarrow \exists y (x \cdot y = 1))$			✓	✓	'Division is possible'
$\forall x (0 < x \rightarrow \exists y (x = y \cdot y))$				✓	'Positive #'s have $\sqrt{\ }$'

- The *theory* of a structure is the set of all sentences it satisfies. Different structures can have the same theory. These are different *models* of the theory.

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Definable sets I

- Given a structure M, a subset of M^n is *definable* if there is a formula specifying it.
- Easy example: The set of pairs of twin primes $(p, p+2) \in \mathbb{N}^2$ is definable by

$$\varphi(x, y) \equiv \ \psi_{\text{prime}}(x) \wedge \psi_{\text{prime}}(y) \wedge y = x + (1+1)$$

  where $\psi_{\text{prime}}(p) \equiv 1 < p \wedge \forall u \forall v (u \cdot v = p \rightarrow u = 1 \vee u = p)$.
- Hard example: The set of pairs $(n, b) \in \mathbb{N}^2$ such that n is a palindrome in base b is definable in $\mathbb{N}$ by some enormous formula only involving 0, 1, $+$, $\cdot$, $<$.

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Definable sets II

- Definable sets in $(\mathbb{N}, 0, 1, +, \cdot, <)$ are very complicated (intimately related to computability theory).
- Definable sets in $(\mathbb{R}, 0, 1, +, \cdot, <)$ are fairly simple: Semi-algebraic sets (Tarski–Seidenberg).
- When the family of definable sets in a structure/theory is *combinatorially tame*, there is often an abstract notion of independence with associated notion of dimension (generalizing linear dimension in vector spaces and transcendence degree in fields).
- Various combinatorial *dividing lines* are studied by model theorists.

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The map of combinatorial tameness

Stable

Algebraically closed fields
Differentially closed fields
Vector spaces
Modules
Free groups*
Curve graphs**

*Sela (2013) **Disarlo, Koberda, de la Nuez González (2020)

The map of combinatorial tameness

NIP	O-minimal: $(\mathbb{R}, +, \cdot, <, \exp)$ $(\mathbb{Q}, +, <)$ Other: $(\mathbb{N}, +, <)$ *p*-adic numbers Alg. closed valued fields	
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Some applications and connections

NIP:

- Application of Pila–Wilkie theorem to the André–Oort conjecture (Pila–Zannier 2008, Pila 2011)
- PAC learnability, finite Vapnik–Chervonenkis dimension (Laskowski 1992), Sauer–Shelah lemma (1972)

Stability:

- Mordell–Lang conjecture for function fields (Hrushovski 1996)
- Online learnability, finite Littlestone dimension (Chase–Freitag 2019)

Continuous model theory

Combinatorial tameness in continuous model theory

Stable	Hilbert spaces Probability algebras Alg. closed valued fields[2] $(\mathbb{Q}_p, +, \lvert x - y \rvert_p)^3$ $\mathbb{R}$-trees[1,3] L^p-lattices ($p < \infty$) Some operator systems Some operator spaces	

[1]Henson and Carlisle (2018) [2]Ben Yaacov (2008, 2009) [3]H. (2020, 2023)

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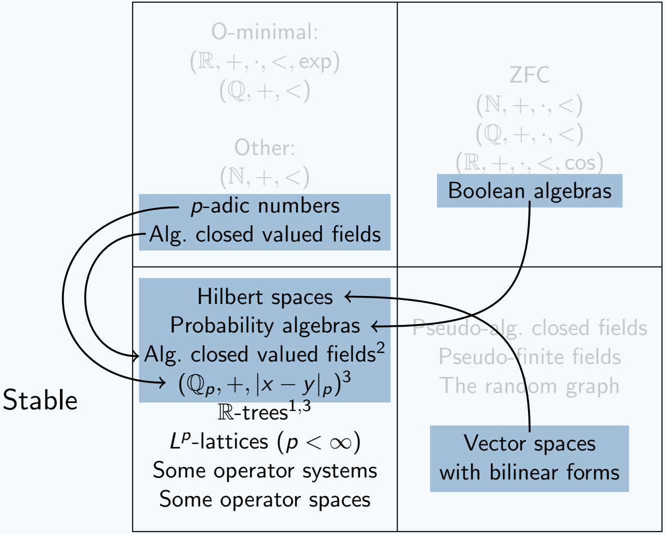

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Combinatorial tameness in continuous model theory

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Continuous first-order logic I

$\mathbb{R}$-valued logic is nearly 100 years old, but model-theory-focused formalism is fairly new (and heavy).

- A *language* is a collection of constant, function, and relation symbols.
- Each function and relation symbol comes with a designated *arity* and *modulus of uniform continuity*.
- Each relation symbol (including the metric) is assigned a bounded interval of possible values. (Could also work with extended metric.)
- *Structures* are complete metric spaces with interpretations of the symbols obeying the specified moduli and bounds.
- *Terms* and *atomic formulas* work as they do in discrete logic.
- Connectives are **all continuous functions** $\mathbb{R}^n \to \mathbb{R}$. Quantifiers are inf and sup.
- Sometimes we also close the collection of formulas under uniformly convergent limits.

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- $\mathbb{R}$-valued *sentences* measure aspects of structures.
- Example: Radius is $\inf_x \sup_y d(x, y)$.
- 'How cold is it?' not 'Is it cold or not?'
- (In)equalities of sentences are called *conditions*:

d is $\{0, 1\}$-valued iff

$$\sup_{xy} \min\{d(x, y), |d(x, y) - 1|\} = 0$$

d is an intrinsic metric iff

$$\sup_{xz} \inf_{y} \max\{|d(x, y) - \tfrac{1}{2}d(x, z)|, |d(y, z) - \tfrac{1}{2}d(x, z)|\} = 0$$

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Definable sets I

- $\mathbb{R}$-valued formulas are the 'correct' generalization of discrete formulas, but sometimes it's useful to have something you can treat more 'discretely.'
- A closed set $D \subseteq M$ is *definable* if the point-set distance

$$d(x, D) = \inf_{y \in D} d(x, y)$$

  is a formula.
- Example: In Hilbert spaces, $B_{\leq r}(a)$ is definable by

$$\varphi(x) = \max\{\|x - a\| - r, 0\}.$$

- Definable sets are characterized by *admitting relative quantification*: For any formula $\varphi(x, y)$, there is a formula equivalent to $\inf_{y \in D} \varphi(x, y)$.

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Definable sets II

Definable sets are nice to have but sometimes hard to find.

Proposition

If D and E are definable sets, then $D \cup E$ is a definable set.

Proof.

$d(x, D \cup E) = \min\{d(x, D), d(x, E)\}$. □

- Complement? Typically not even closed.
- Intersection? Unclear: $d(x, D \cap E) \neq \max\{d(x, D), d(x, E)\}$.

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Dictionaric theories

Continuous theories are easier to work with when they have 'enough' definable sets.

Definition

A theory is *dictionaric* if for every formula $\varphi(\bar{x})$ (with parameters) and every $r < s$, there is a definable set D such that $\{\varphi \leq r\} \subseteq D \subseteq \{\varphi < s\}$.

All discrete theories are dictionaric when considered as continuous theories.

Theorem (H.)

ω-stable theories and randomizations of arbitrary (discrete or continuous) theories are dictionaric.

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If T is dictionaric, then for any definable sets D and E, there are definable F 'arbitrarily close' to E such that $D \cap F$ is definable.

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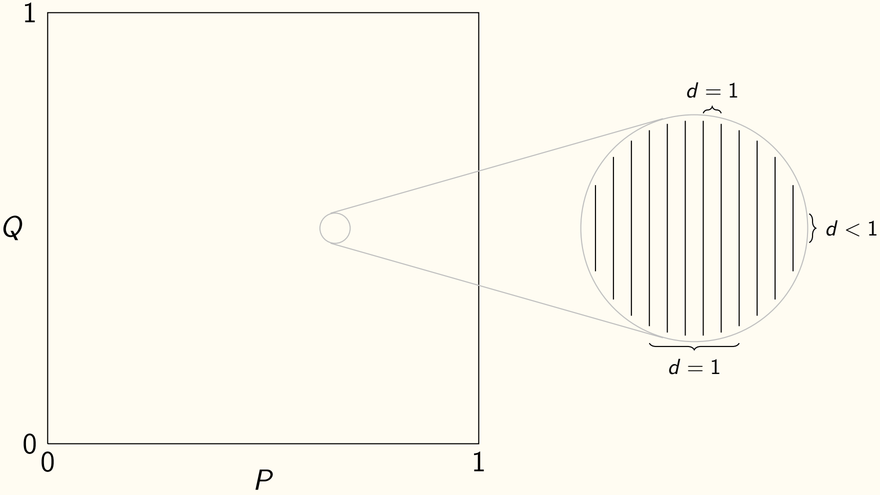
What can go wrong?

Example: $\{P, Q\}$-structure with universe $[0,1]^2$



$P((x,y)) = x$
$Q((x,y)) = y$

$d(z,w) = |Q(z) - Q(w)|$ if $P(z) = P(w)$.
$d(z,w) = 1$ otherwise.

Definable sets in the example

- Any formula $\varphi(z)$ (without parameters) is equivalent to $f(P(z), Q(z))$ for some continuous $f : [0,1]^2 \to \mathbb{R}$. (Continuity is with regards to the standard compact topology.)
- A closed set $F \subseteq [0,1]^2$ is definable iff F is in the interior of $F^{<\varepsilon}$ for every $\varepsilon > 0$. Specific case of a general characterization of definable sets (interplay between two topologies).
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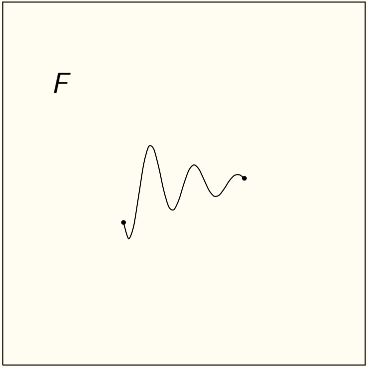

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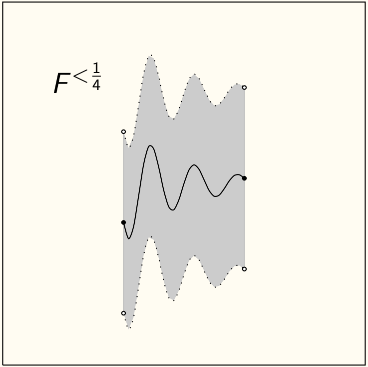

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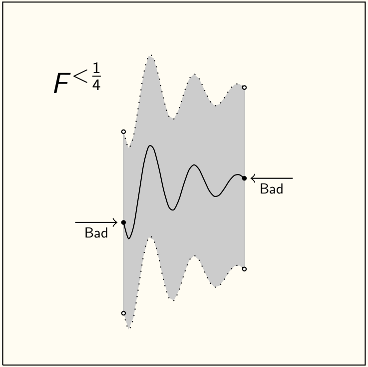

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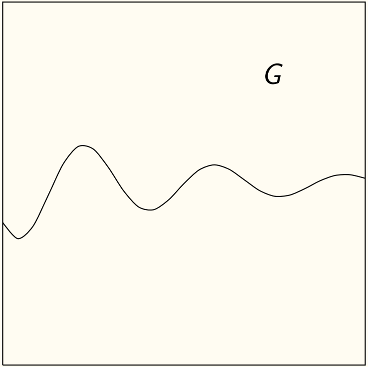

Definable sets in the example

- Any formula $\varphi(z)$ (without parameters) is equivalent to $f(P(z), Q(z))$ for some continuous $f : [0,1]^2 \to \mathbb{R}$. (Continuity is with regards to the standard compact topology.)
- A closed set $F \subseteq [0,1]^2$ is definable iff F is in the interior of $F^{<\varepsilon}$ for every $\varepsilon > 0$. Specific case of a general characterization of definable sets (interplay between two topologies).
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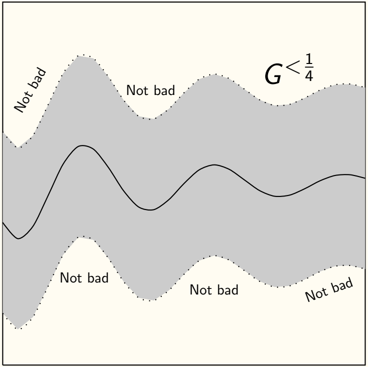

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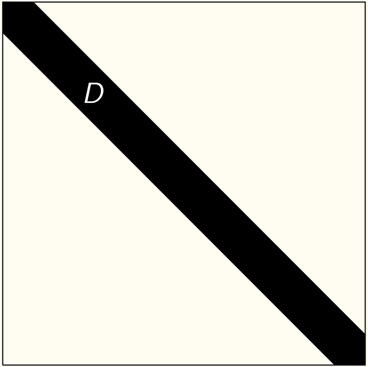

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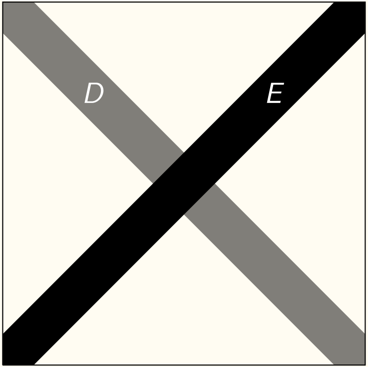

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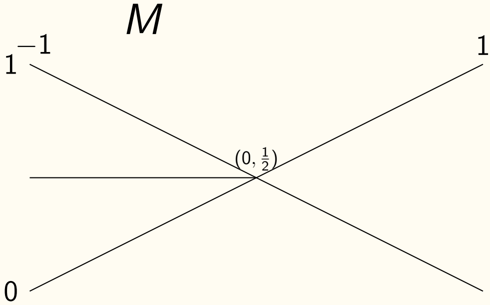

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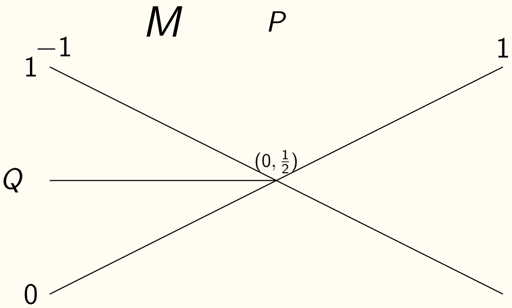

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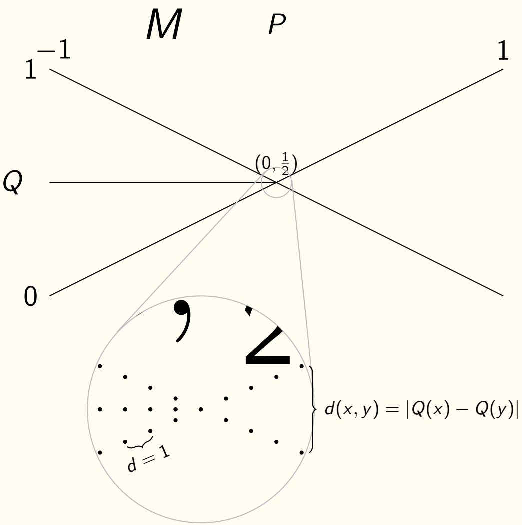

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- $d(x, y) = |Q(x) - Q(y)|$ if $P(x) = P(y)$ and is 1 otherwise.
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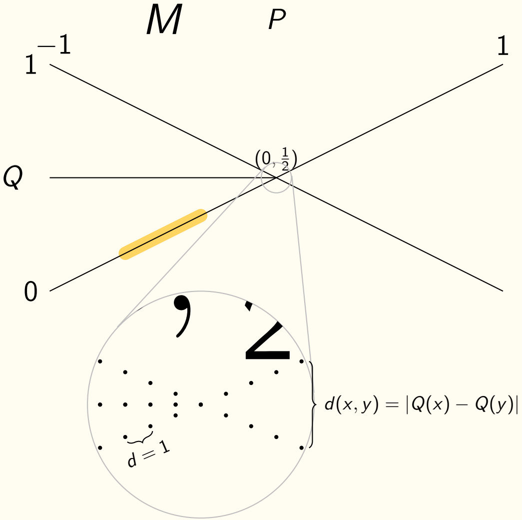

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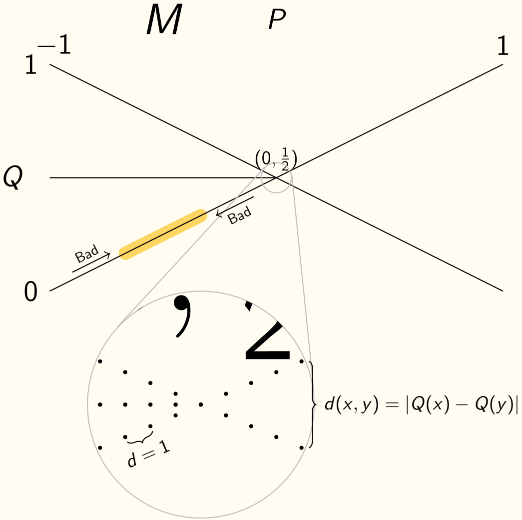

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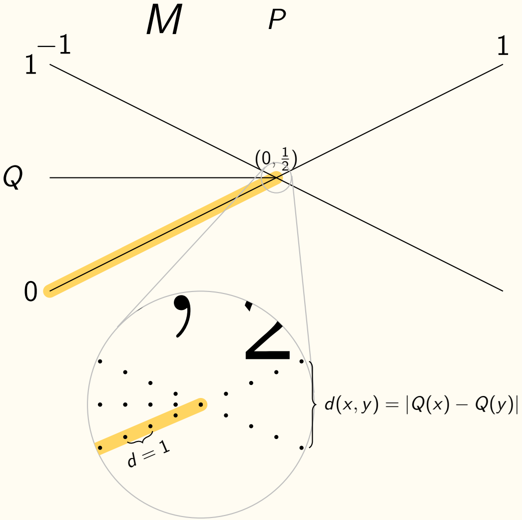

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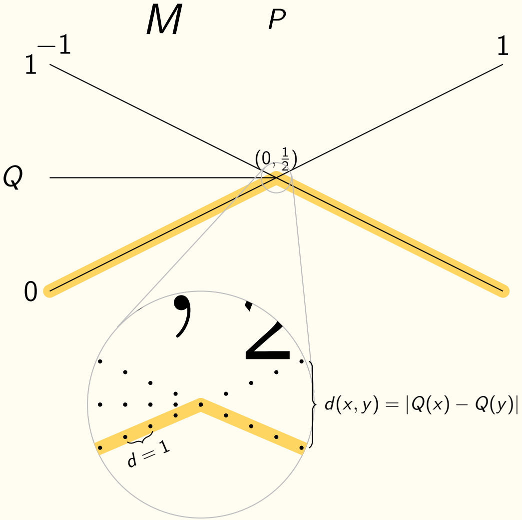

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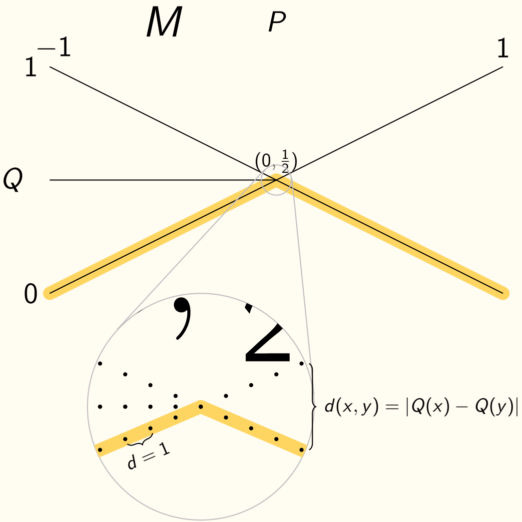

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- Has precisely 22 definable sets.

Which finite semilattices can we have?

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Every finite semilattice is the semilattice of definable sets (in one variable) for some complete superstable theory.

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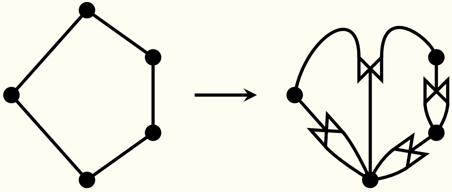
Proof sketch.

Construct a configuration that can act as an AND gate. Glue these together to build a 'logic circuit' that directly codes the diagram of a given finite semilattice. □



Structure with N_5 as its semilattice of definable sets.

Thank you

The map of combinatorial tameness

Stable

Algebraically closed fields
Differentially closed fields
Vector spaces
Modules
Free groups*
Curve graphs**

*Sela **Disarlo, Koberda, de la Nuez González

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Dependent (NIP)	O-minimal: $(\mathbb{R}, +, \cdot, <)$ $(\mathbb{R}, +, \cdot, <, \exp)$ $(\mathbb{Q}, +, <)$ Other: $(\mathbb{N}, +, <)$ *p*-adic numbers Alg. closed valued fields	
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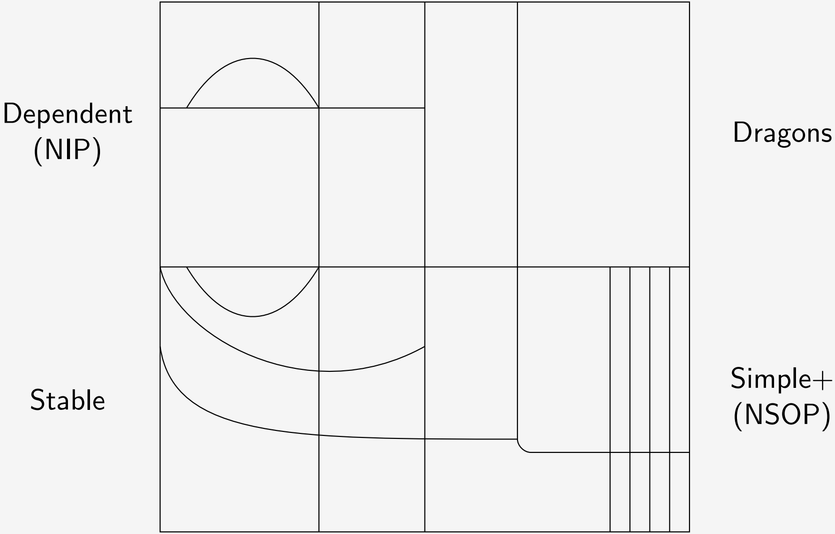

Conant, forkinganddividing.com

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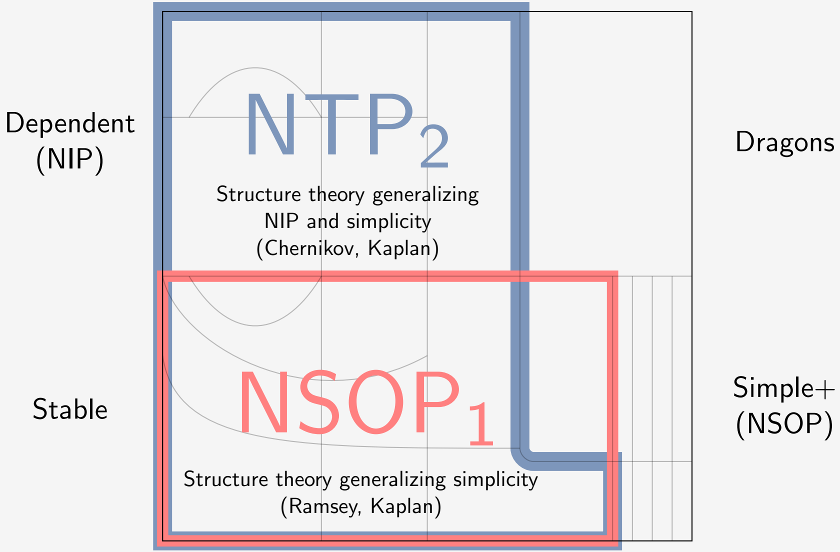

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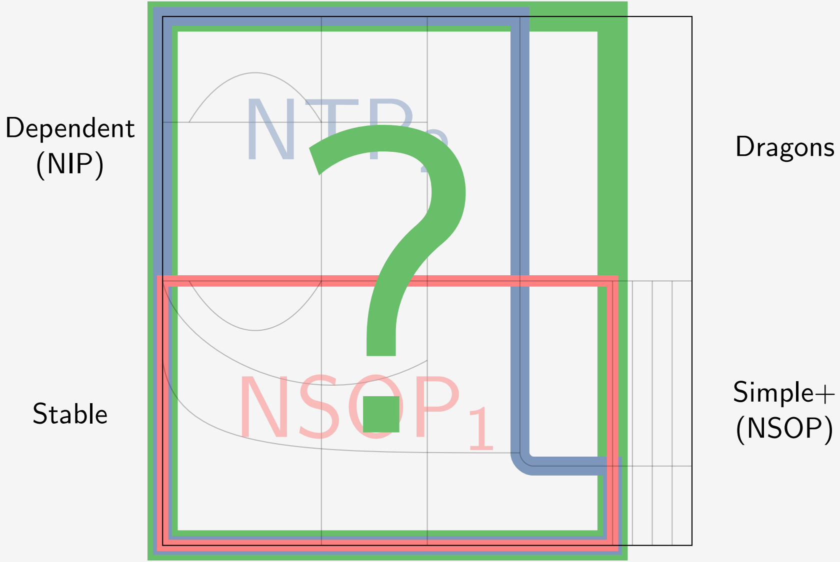
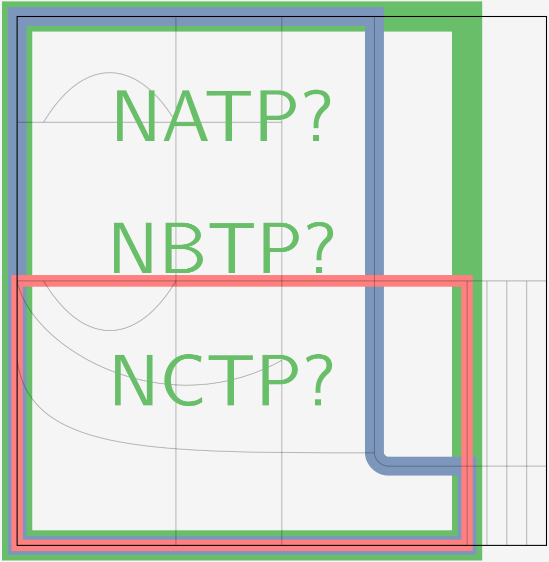


The map of combinatorial tameness

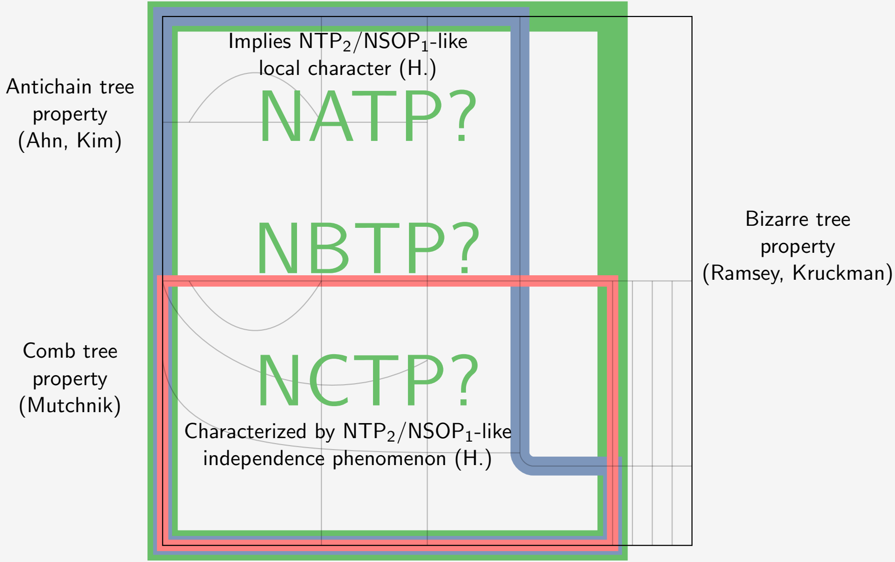
Antichain tree property (Ahn, Kim)

Comb tree property (Mutchnik)



Bizarre tree property (Ramsey, Kruckman)

The map of combinatorial tameness



Some applications and connections

NIP:

- IAS *Special Year on Arithmetic Geometry, Hodge Theory, and o-minimality* (2025-26)
- Application of Pila–Wilkie theorem to the André–Oort conjecture (Pila–Zannier 2008, Pila 2011)
- PAC learnability, finite Vapnik–Chervonenkis dimension (Laskowski 1992), Sauer–Shelah lemma (1972)
- Rosenthal compacta (Bourgain–Fremlin–Talagrand 1978, Simon 2014)

Stability:

- Mordell–Lang conjecture for function fields (Hrushovski 1996)
- Online learnability, finite Littlestone dimension (Chase–Freitag 2019)
- Connections with geometric group theory (Sela 2001-2013, Disarlo–Koberda–de la Nuez González 2020)

General:

- Ax–Grothendieck (1966/8), Ax–Kochen (1965)

Continuous model theory

- Real-valued logic goes back to the early 20th century:

J. Łukasiewicz und A. Tarski.

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- Took a long time for stability theory and neo-stability theory to be applied.

Characterization of dictionaricity

Theorem (H.)

The following are equivalent:

1. $S_n(T)$ is dictionaric.
2. Definable sets separate disjoint closed subsets of $S_n(T)$.
3. For every disjoint closed $F, G \subseteq S_n(T)$, there is a definable set D such that either $F \subseteq D$ and $D \cap G = \varnothing$ or $G \subseteq D$ and $D \cap F = \varnothing$.
4. $S_n(T)$ has a network of definable sets (i.e. for every $p \in U \subseteq S_n(T)$, there is a definable set D such that $p \in D \subseteq U$).
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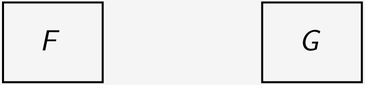

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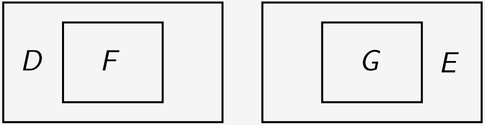

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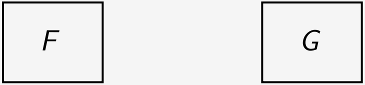

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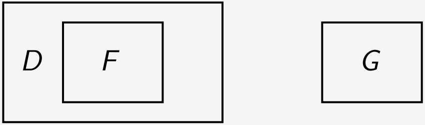

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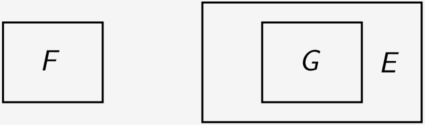

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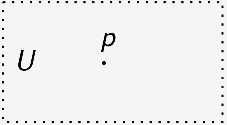

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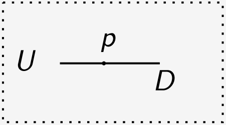

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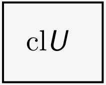

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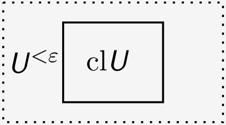
5

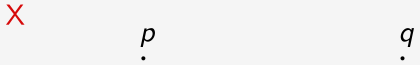


Characterization of dictionaricity

Theorem (H.)

The following are equivalent:

1. $S_n(T)$ is dictionaric.
2. Definable sets separate disjoint closed subsets of $S_n(T)$.
3. For every disjoint closed $F, G \subseteq S_n(T)$, there is a definable set D such that either $F \subseteq D$ and $D \cap G = \varnothing$ or $G \subseteq D$ and $D \cap F = \varnothing$.
4. $S_n(T)$ has a network of definable sets (i.e. for every $p \in U \subseteq S_n(T)$, there is a definable set D such that $p \in D \subseteq U$).
5. For every $\varepsilon > 0$, $S_n(T)$ has a basis of open sets U satisfying $\mathrm{cl}U \subseteq U^{<\varepsilon}$.



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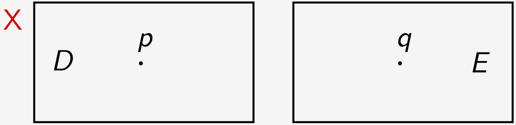
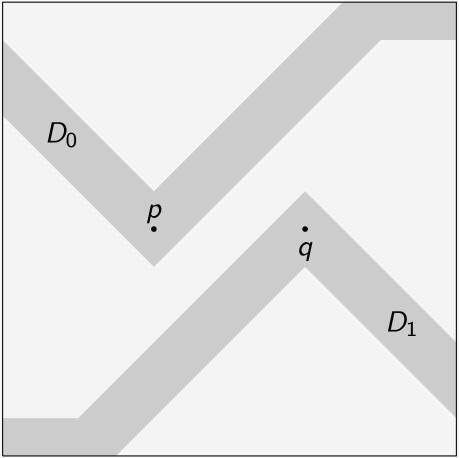
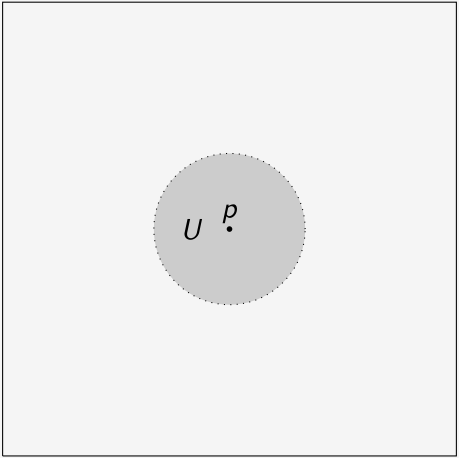


'Hausdorff' is not quite enough I



Almost any two points are separated by disjoint definable neighborhoods.

'Hausdorff' is not quite enough II



There is no non-empty definable D with $D \subseteq U$.