How bad could it be?
The semilattice of definable sets in continuous logic

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University of Maryland Logic Seminar

Continuous logic

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- Elementary extensions: $\mathbb{R} \oplus \mathbb{Q}^{\oplus\kappa}$, where $\mathbb{Q}^{\oplus\kappa}$ has $\{0, 1\}$-metric.

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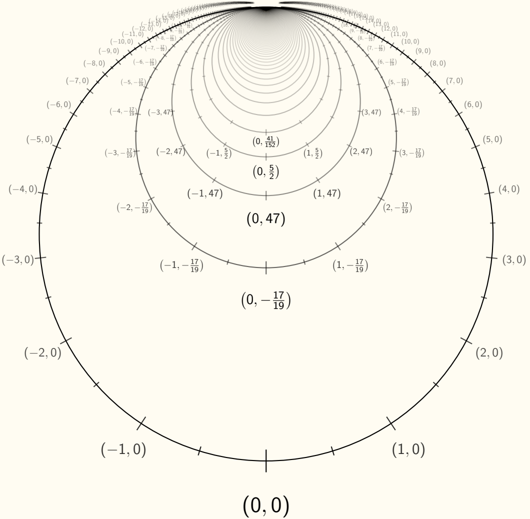
- (H.) Any compact topometric space (X, τ, ρ) with open metric ρ is isomorphic to $S_1(T)$ for some strictly stable T.

$S_1(\mathbb{R} \oplus \mathbb{Q})$

Unique non-algebraic type



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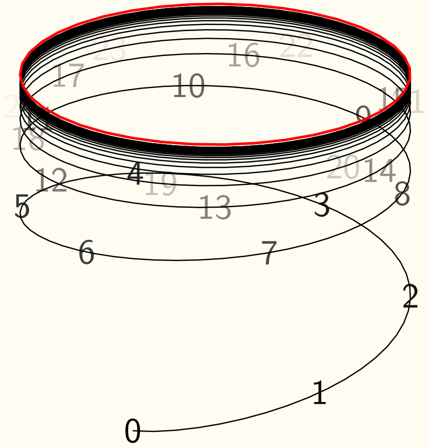
- D is definable iff $D^{<r}$ is open for every $r > 0$.
- If T is ω-stable, then $S_n(A)$ always has a basis of definable neighborhoods. (T is *dictionaric*.)

Many definable sets: $S_1(M)$, $M = (\mathbb{R}_{\geq 0}, \cos, \sin, d)$



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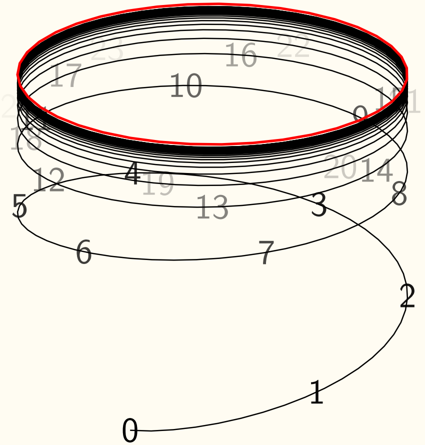

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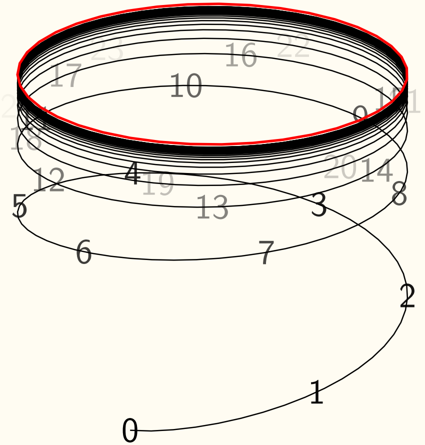
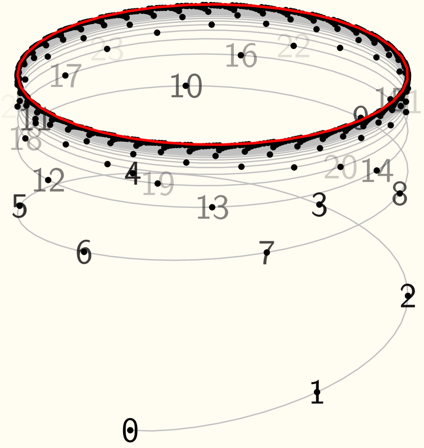


- $\mathrm{Th}(M)$ is ω-stable.
- Metric on non-algebraic types is (roughly) path metric.

Few definable sets: $S_1(N)$, $N = (\mathbb{N}, \mathrm{succ}, \cos, \sin, d)$



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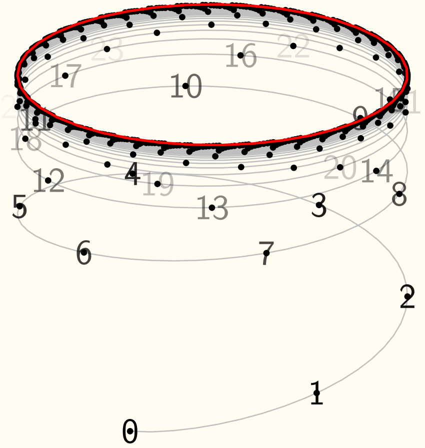

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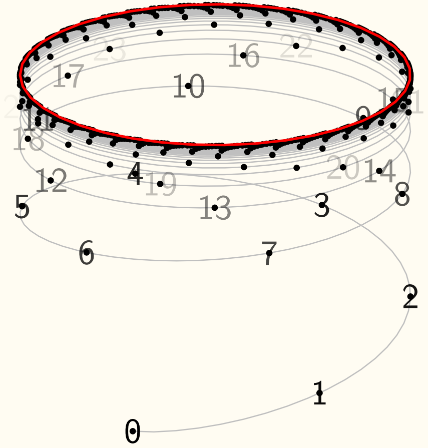

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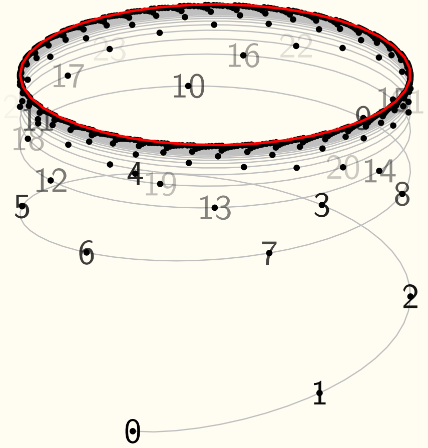
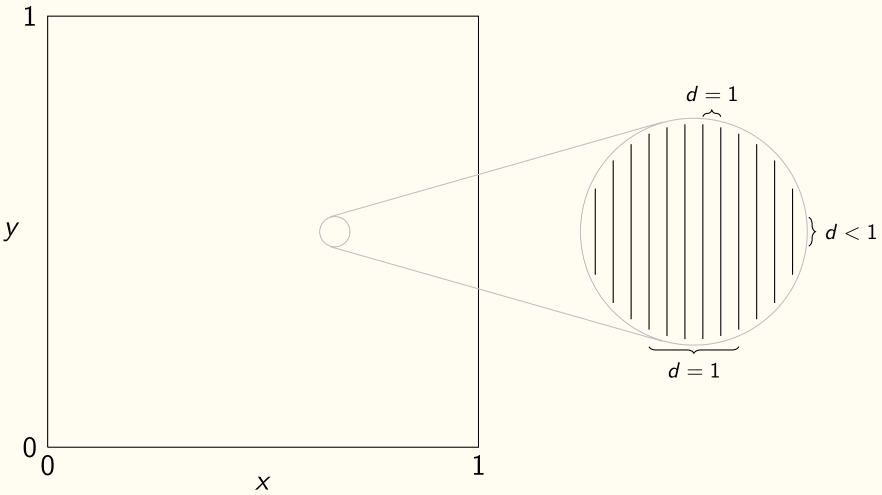


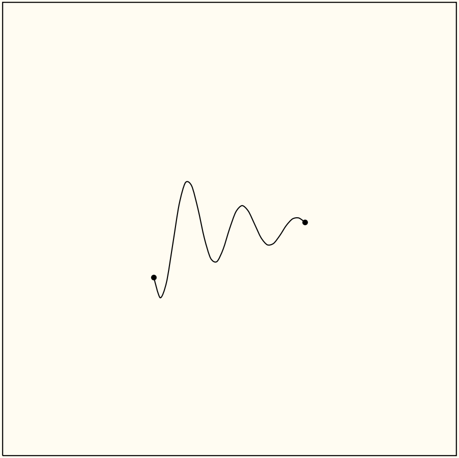
- $\mathrm{Th}(N)$ is superstable.
- Metric on non-algebraic types is discrete. Every definable set is either finite and algebraic or cofinite and co-algebraic.

Many but not enough I



$$d((x_0, y_0), (x_1, y_1)) = 1 \text{ if } y_0 \neq y_1.$$
$$d((x_0, y), (x_1, y)) = |x_0 - x_1|.$$

Many but not enough II



Closed F,

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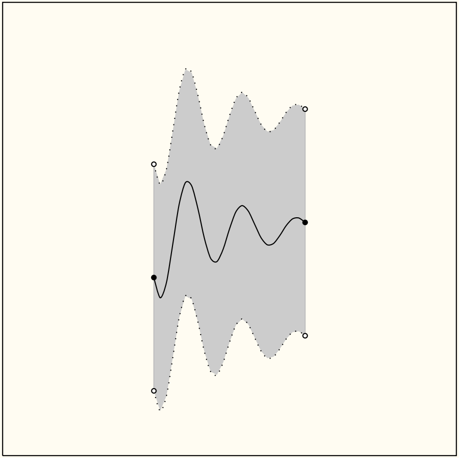

Closed F, with $F^{<\frac{1}{4}}$.

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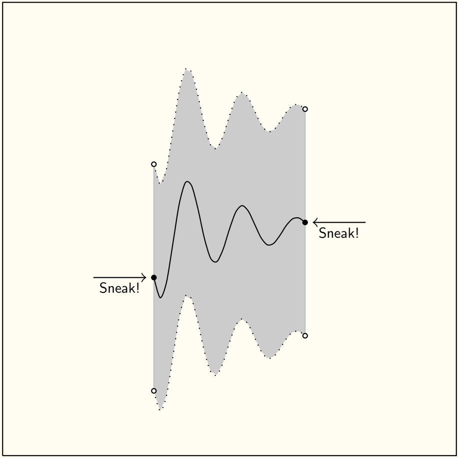

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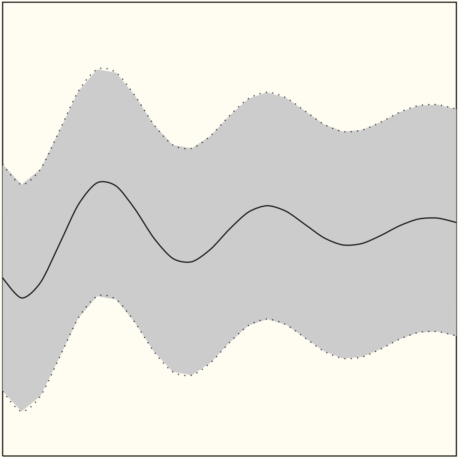
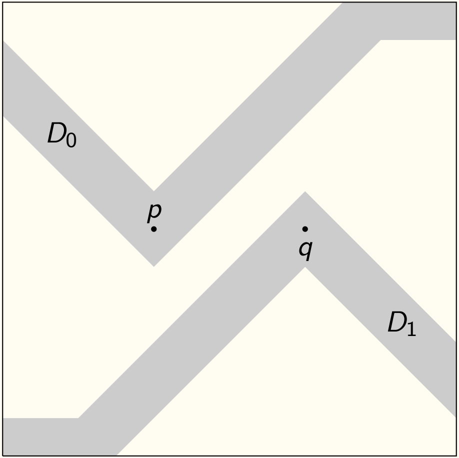
Not definable.

Many but not enough III



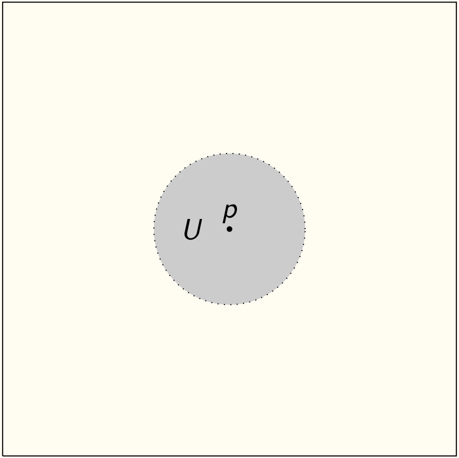
Definable set D, with $D^{<\frac{1}{4}}$.

Many but not enough IV



Almost any two points are separated by disjoint definable neighborhoods.

Many but not enough V



There is no non-empty definable D with $D \subseteq U$.

The semilattice of definable sets

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Proposition

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Given a type space $S_n(T)$, the collection of definable subsets of it forms a bounded upper semilattice ($\varnothing$ and $S_n(T)$ are always definable) under unions.

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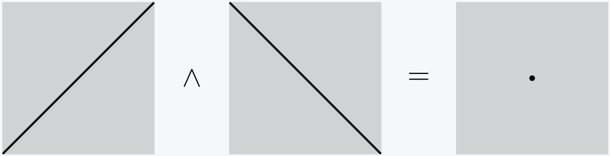
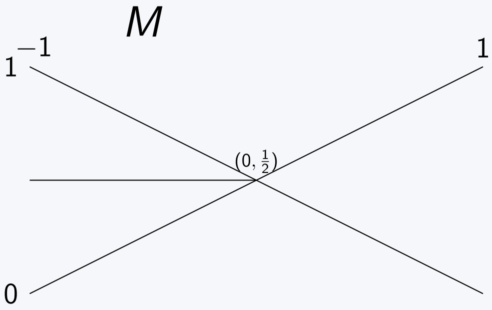
In square type space:



Prototypical example: Structure



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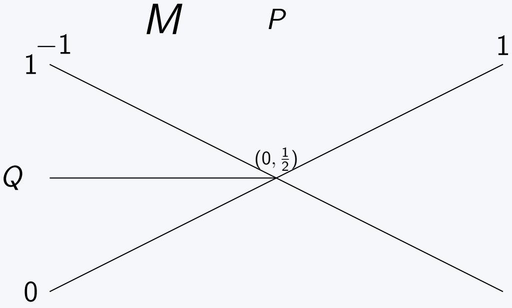

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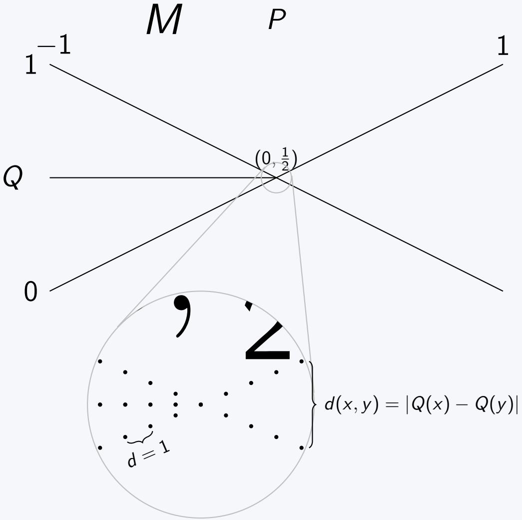

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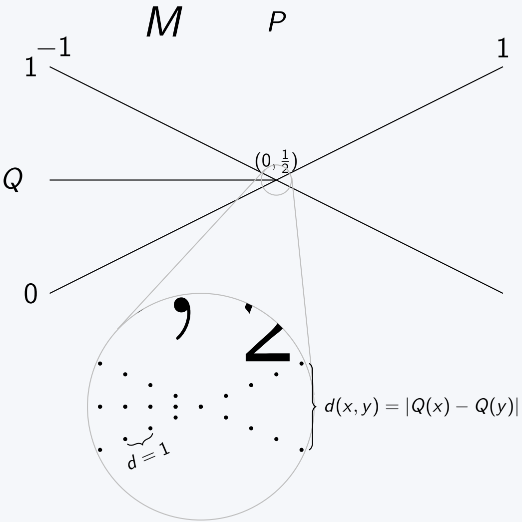

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- $d(x, y) = |Q(x) - Q(y)|$ if $P(x) = P(y)$ and is 1 otherwise.
- Let $T = \mathrm{Th}(M)$.

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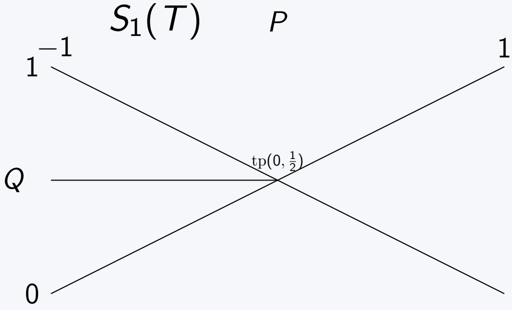

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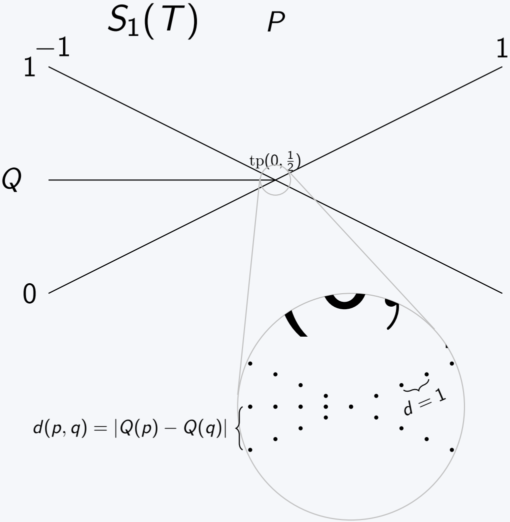

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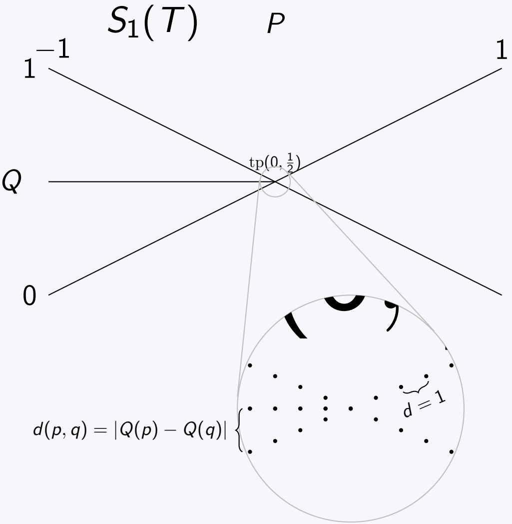

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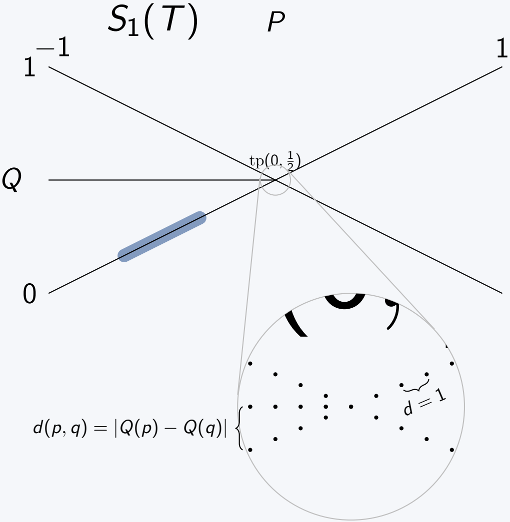

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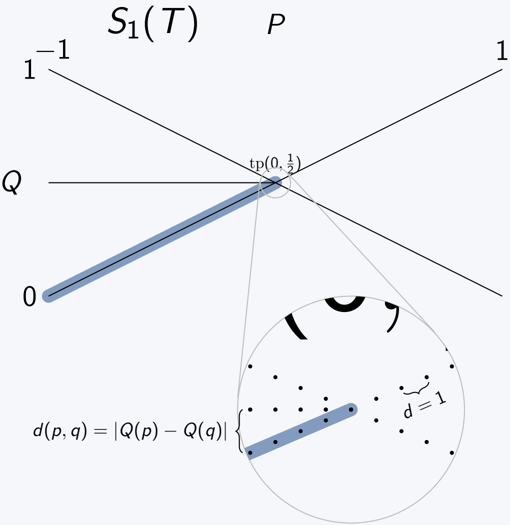

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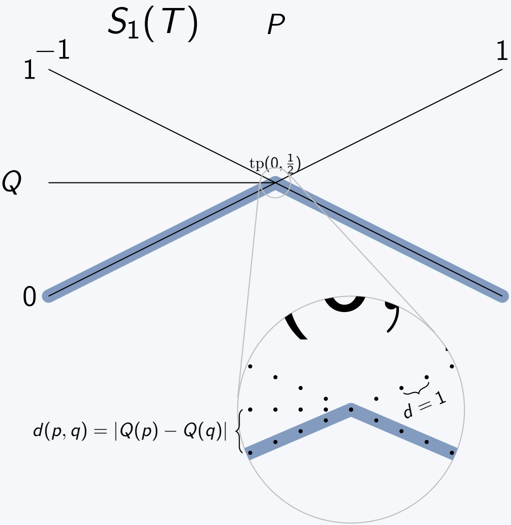

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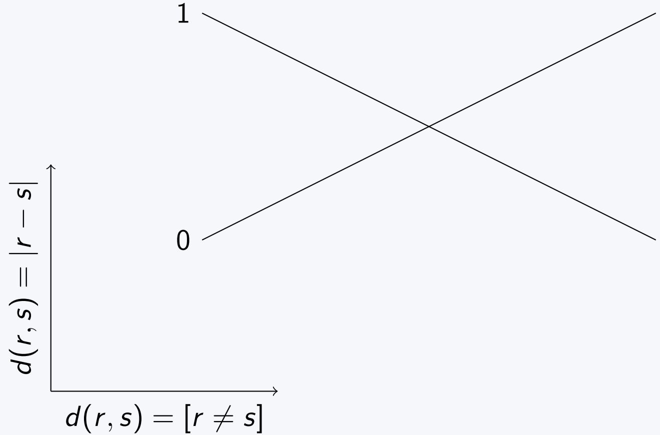

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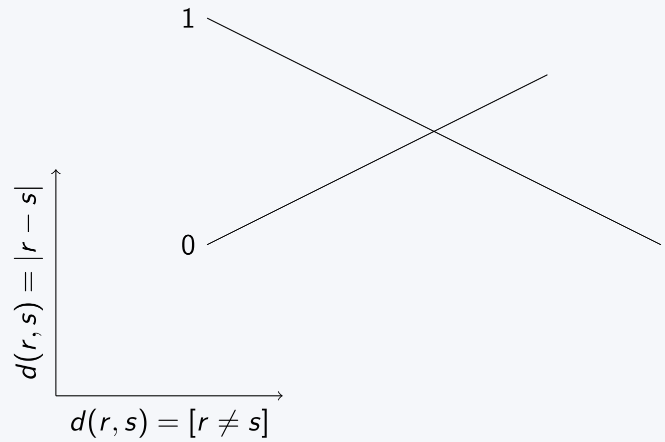

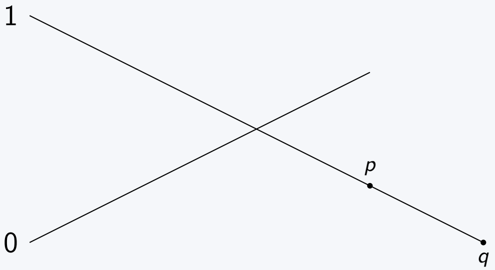

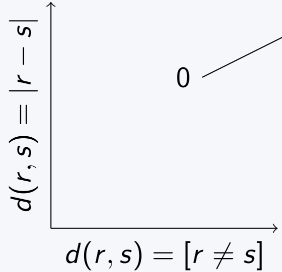

$$p(x) \models \inf_y |d(x,y) - \tfrac{1}{2}| = 0$$

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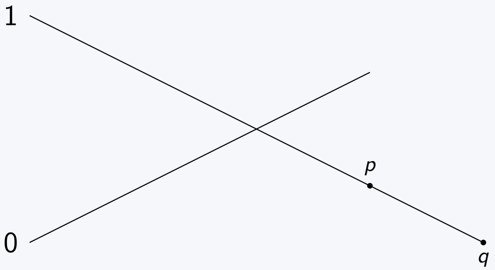
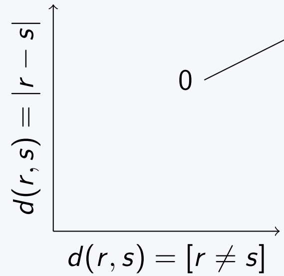
Wires





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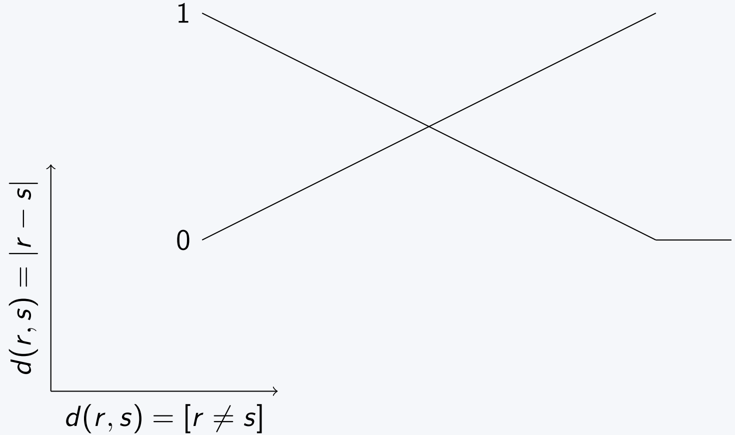

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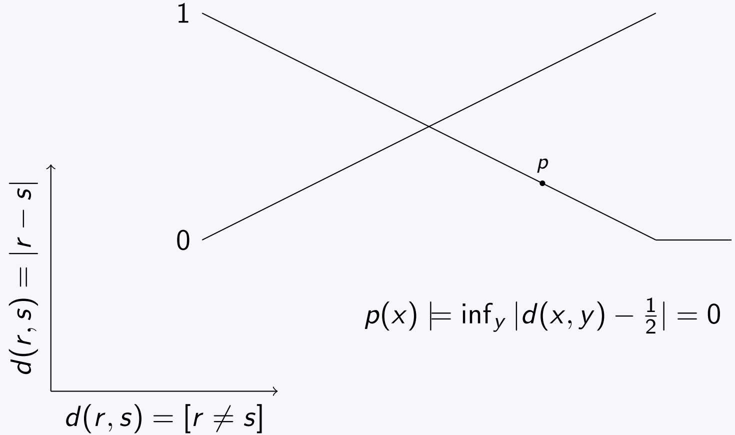

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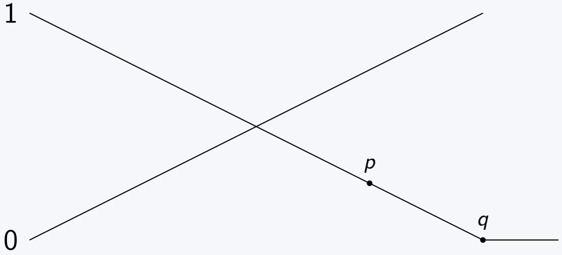

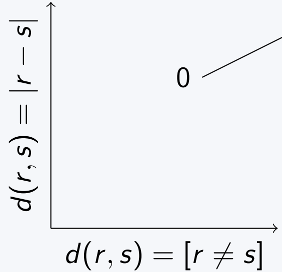

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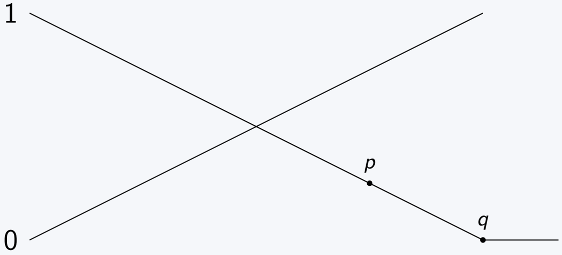
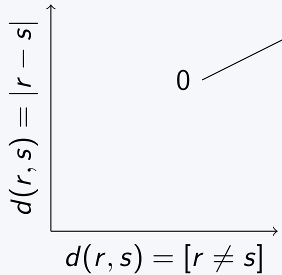
Wires





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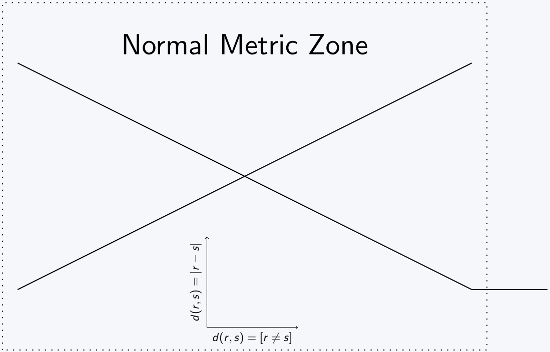
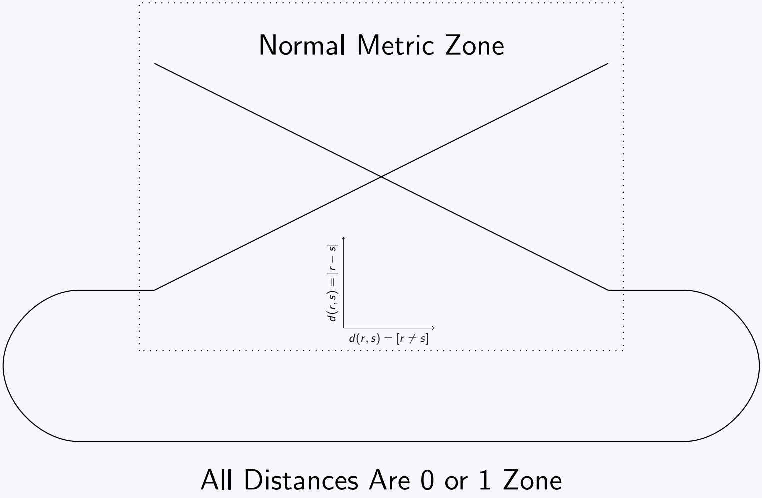
Wires



All Distances Are 0 or 1 Zone

Wires



Normal Metric Zone
$d(r,s) = |r - s|$
$d(r,s) = [r \neq s]$
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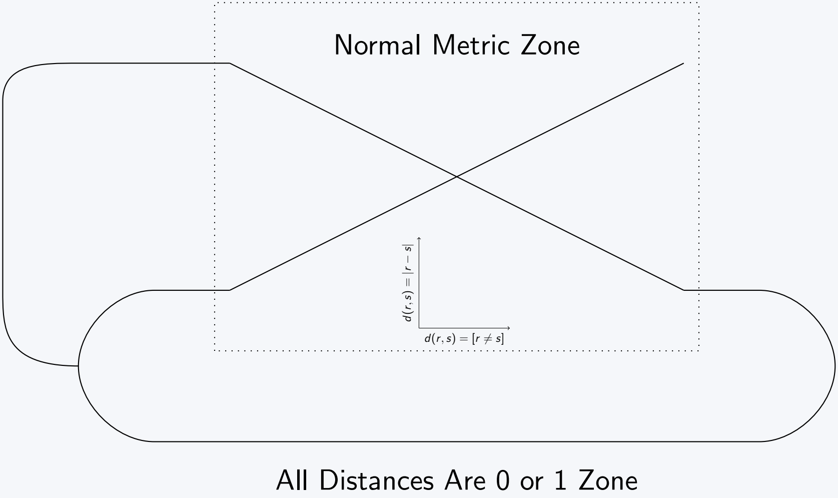

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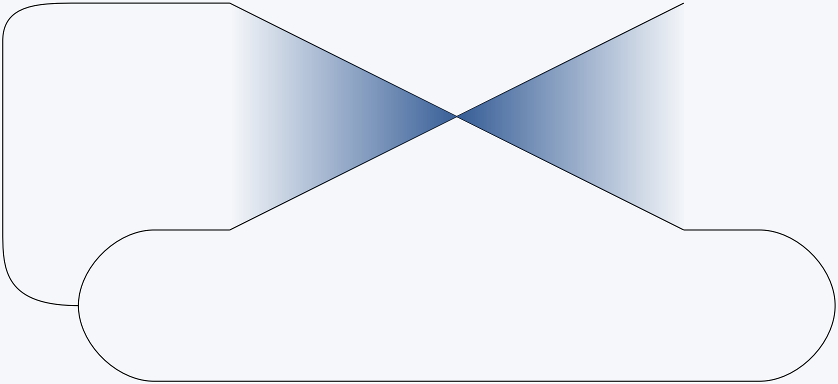

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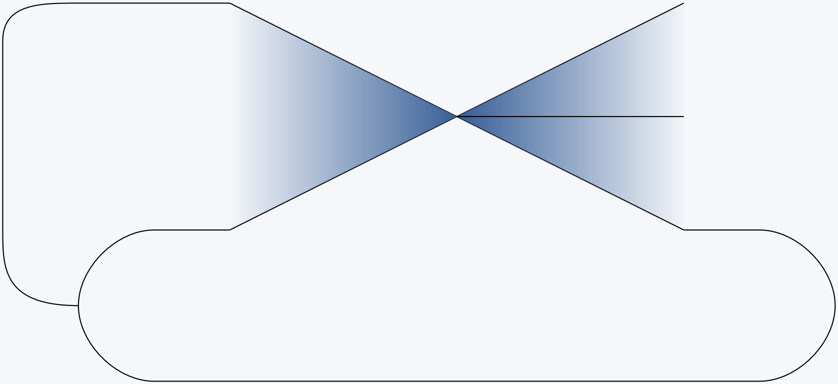

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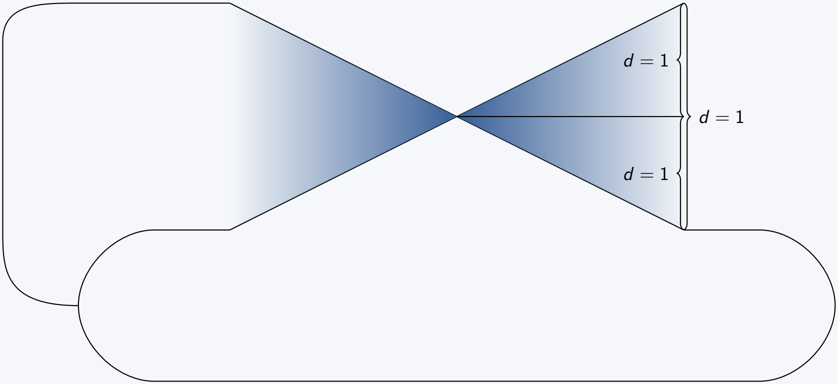
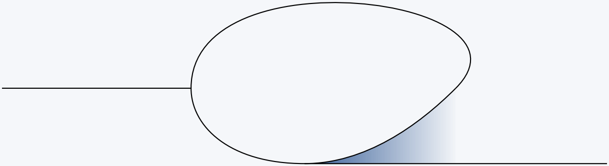
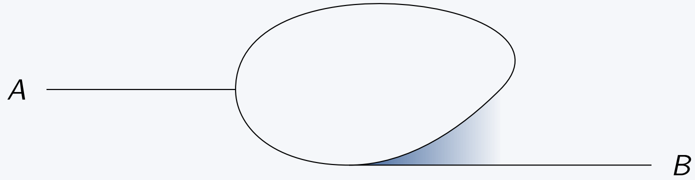


A diode



A diode

- Open Set U



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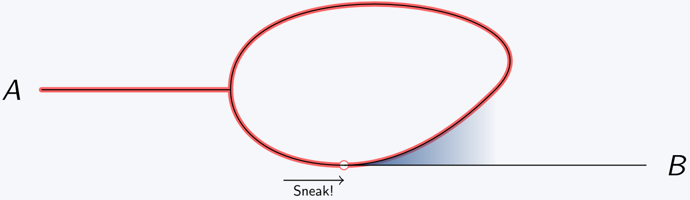

$S_1(T) \setminus U$ is not definable. X

A diode

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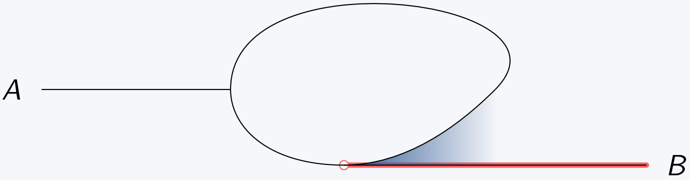

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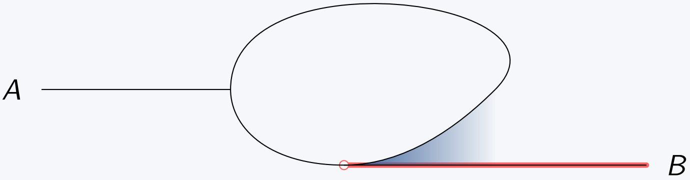
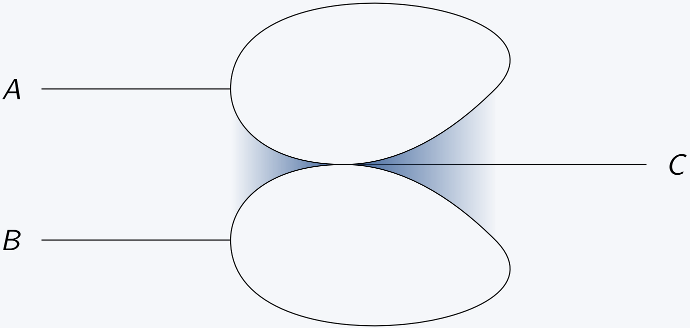


Roughly: $S_1(T) \setminus U$ is definable iff $A \in U \to B \in U$.

An AND gate

- Open Set U



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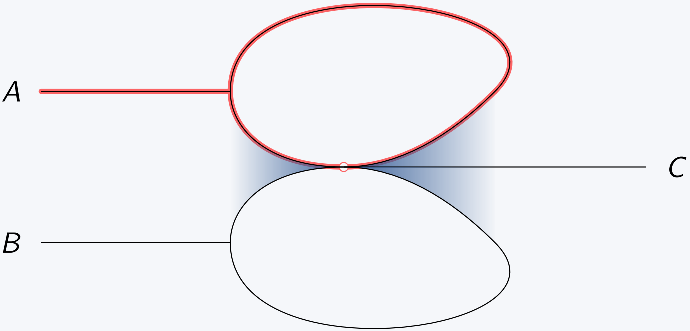

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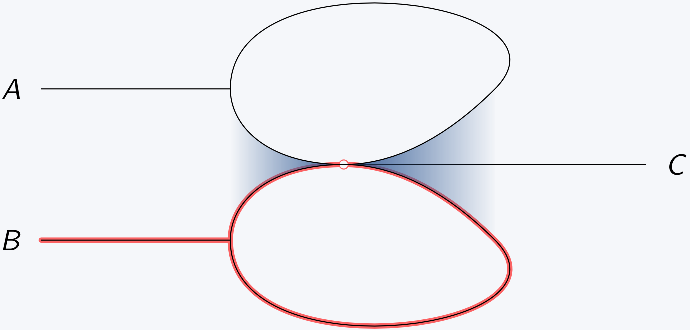

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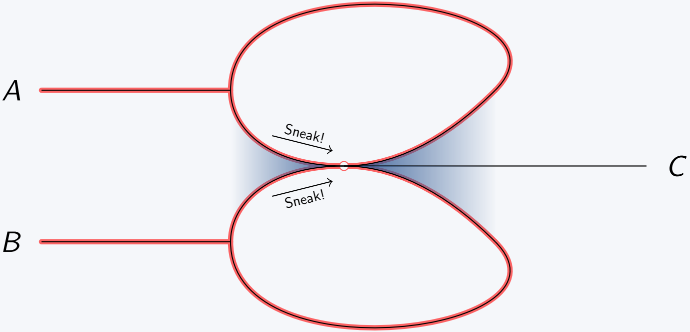

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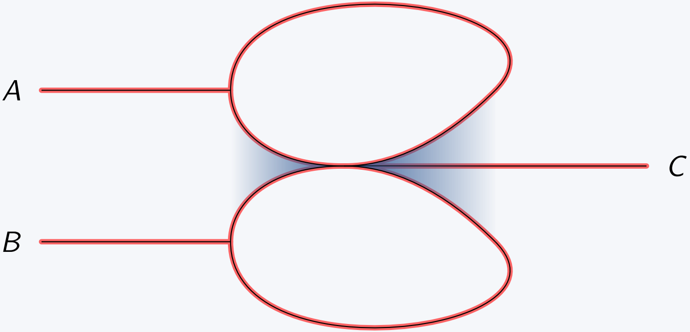

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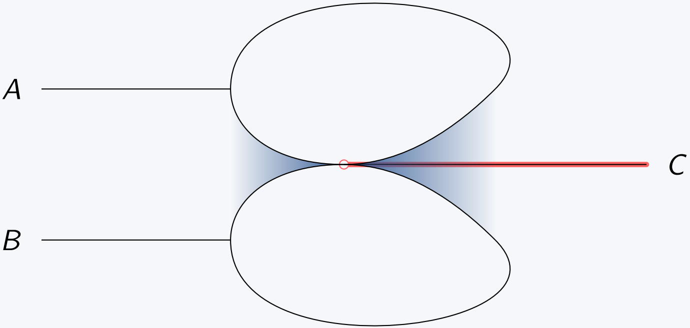

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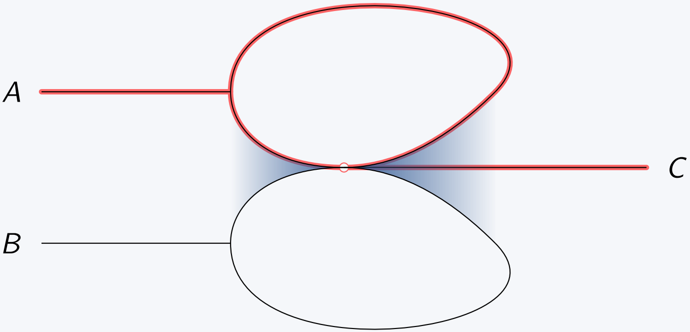

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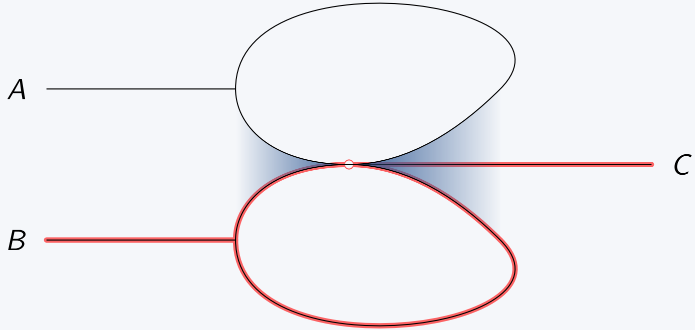
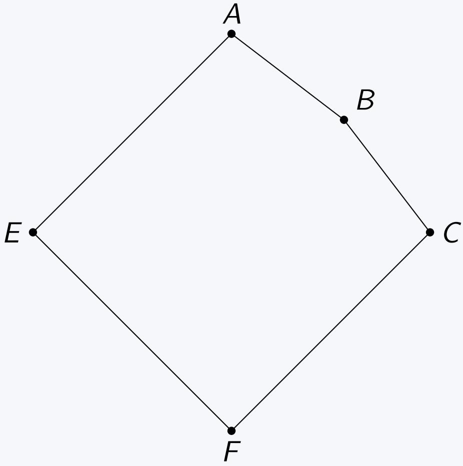


Roughly: $S_1(T) \setminus U$ is definable iff $A \in U \wedge B \in U \to C \in U$.

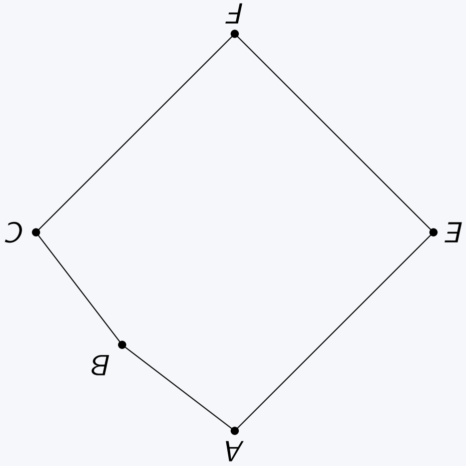
The construction

Take your favorite finite lattice with more than one element



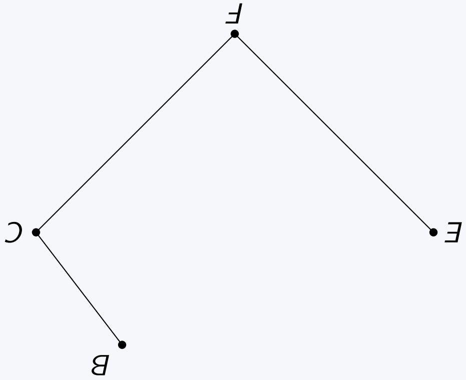
The construction

Take your favorite finite lattice with more than one element and flip it upside down.



The construction

Take your favorite finite lattice with more than one element and flip it upside down. Also cut the bottom off (point representing $\varnothing$ would be redundant).



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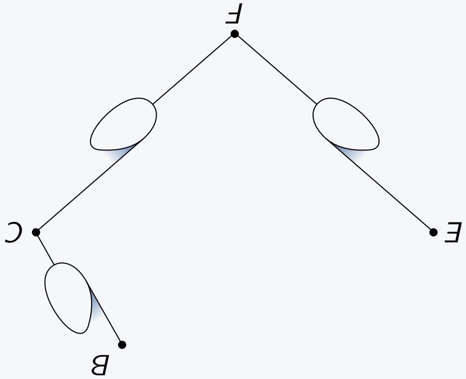
Diode for each pair with $x \leq y$



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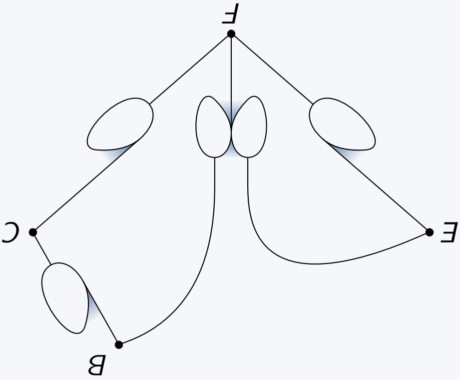
and AND gate for each triple with $x \leq y \vee z$



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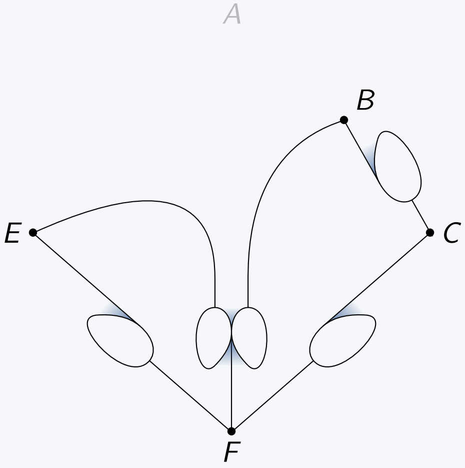

Each definable set is uniquely specified by the unique minimal node that it does not contain. (Including A.)

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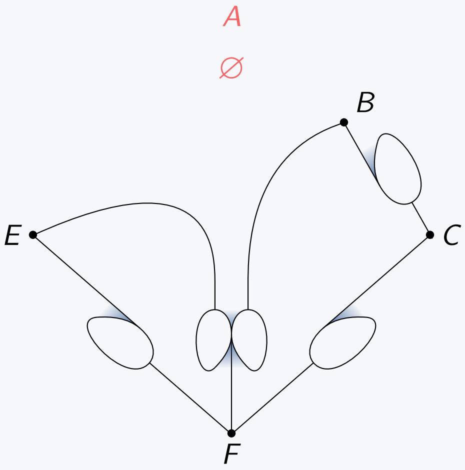

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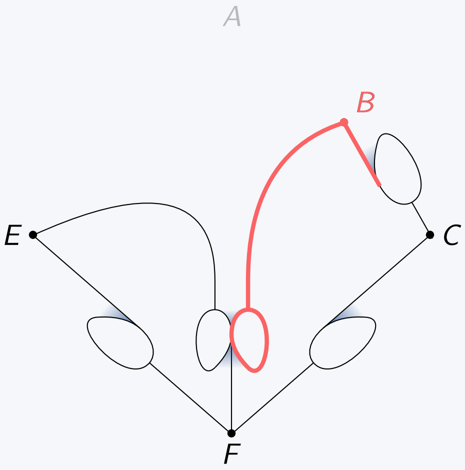

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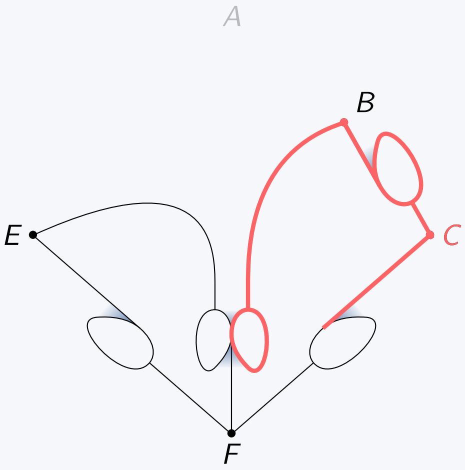

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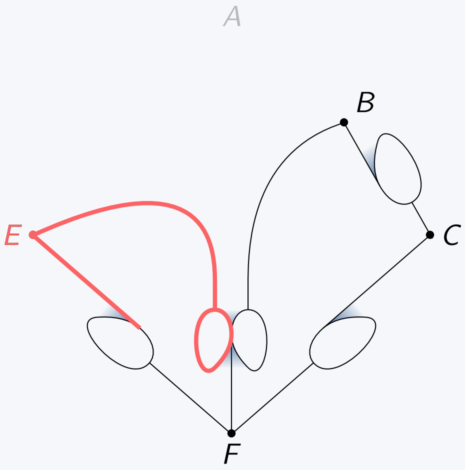

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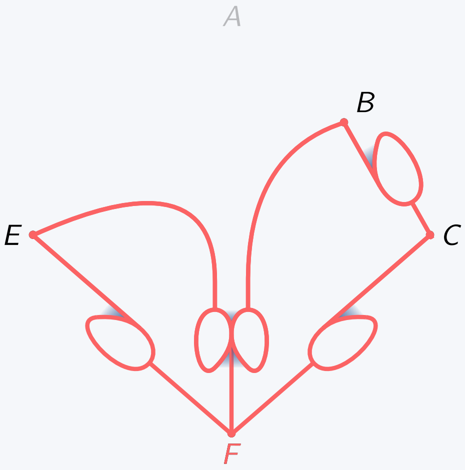

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Superstability

- When you interpret a type space X built this way as a structure, the resulting theory's $S_1(T)$ *is* X. (X is *autological*.)

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Question

Which type spaces are autological?

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- Resulting theory is weakly minimal with trivial geometry, so superstable.

Question

Which type spaces are autological? Is the theory of an autological type space always weakly minimal with trivial geometry?

To infinity and not very much further

Which infinite semilattices can we have?

- General principles tell us that a type space (in a countable language) must have either $\leq \aleph_0$ or $2^{\aleph_0}$ definable sets. (Complete metric space.)

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- There is a way to 'compactify' infinite but locally finite graphs of the kind we built here to get the associated lattice together with a new bottom element, but not every countable lattice can be expressed in this way (e.g. $\omega + 2$).

Thank you