Strongly Minimal Sets in Continuous Logic

James Hanson

University of Wisconsin-Madison

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Uncountably Categorical Theories

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A theory is *κ-categorical* if it has a unique model of cardinality κ.

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These ingredients give you: A set with a good dimension theory (strongly minimal, from ω-stable) that 'controls' everything (no Vaughtian pairs).

Continuous Logic

- Generalization of first-order logic for *metric structures*: Complete bounded metric spaces with uniformly continuous $\mathbb{R}$-valued predicates.
- Quantifiers are $\sup$ and $\inf$. Connectives are arbitrary continuous functions $F : \mathbb{R}^k \to \mathbb{R}$ for $k \leq \omega$. (In this talk: No distinction between formula and definable predicate. More permissive but equivalent.)
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Definition

A zeroset or type is *algebraic* if it is metrically compact in every model.

These are precisely the sets that do not grow in elementary extensions.

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Inseparably categorical theories are ω-stable (count types with metric density character).

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Converse?

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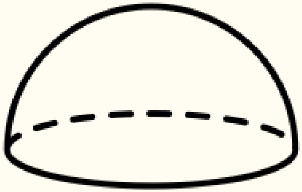
- ...does not have any strongly minimal types (see picture).



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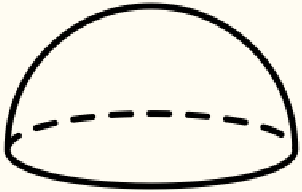
- IHS does not even interpret a strongly minimal theory.



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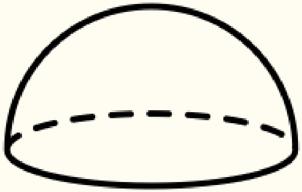
- ...does not have any strongly minimal types (see picture).
- IHS does not even interpret a strongly minimal theory.
- So, let's just move the goalposts and assume we can find strongly minimal types.



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Moving the goalposts: Inseparable categoricity in the presence of strongly minimal types

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- Has a unique non-algebraic type over any parameters.
- If p, a type over A, has a unique non-forking extension q, a type over $B \supseteq A$, such that q is the unique non-algebraic type in a B-definable strongly minimal set E, can we always find an A-definable strongly minimal set D such that p is the unique non-algebraic type in D? (Note you can always do this in discrete logic.)

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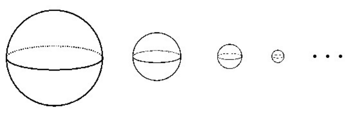

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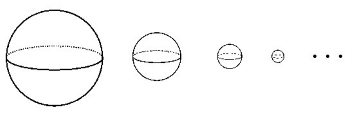

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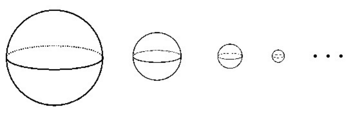

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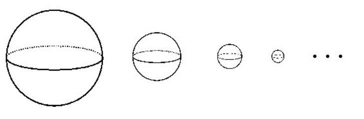

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What's the problem?

Not enough definable sets

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- In a dictionaric theory with no Vaughtian pairs, minimal sets are strongly minimal. (Same for approximately (strongly) minimal pairs.)

Partial Baldwin-Lachlan Condition

Theorem (H.)

For every $n \leq \omega$ there is an inseparably categorical theory with a $\varnothing$-definable strongly minimal imaginary I such that $dim(I)$ can be anything $\leq \omega$ but $S_1(\mathfrak{A})$ has a strongly minimal type iff $\dim(I(\mathfrak{A})) \geq n$.

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Theorem (H.)

A theory with a minimal set (resp. imaginary) over the prime model is inseparably categorical iff it is dictionaric and has no (imaginary) Vaughtian pairs.
Such a theory has $\leq \aleph_0$ separable models and if it has a $\varnothing$-definable approximately minimal pair then it has 1 or $\aleph_0$ separable models.

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Which, of course, raises the question:

When can we find strongly minimal types?

Two Axes of Difficulty

Continuous logic introduces two new difficulties:

- Lack of local compactness of models.
- Lack of total disconnectedness of type spaces.

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If T has a locally compact model, then it is inseparably categorical iff it is ω-stable and has no Vaughtian pairs.
Such a theory has $\leq \aleph_0$ many separable models.

Totally Disconnected Type Spaces/Ultrametric Theories

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A theory has totally disconnected type spaces iff it is dictionaric and has a $\varnothing$-definable ultrametric with scattered distance set and equivalent to the metric.
Such theories are bi-interpretable with many-sorted discrete theories.

Not all ultrametric theories are dictionaric.

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A literal translation of the Baldwin-Lachlan condition fails in continuous logic.

Thank you

Why can't you define strongly minimal in terms of definable sets?

- There is a strictly superstable theory with $2^{\aleph_0}$ many distinct non-algebraic types over any parameter set but for which every pair of disjoint definable sets at most one is non-compact.
- D is Strongly minimal is equivalent to: D is dictionaric and for every pair of disjoint definable subsets of D at most one is non-algebraic.

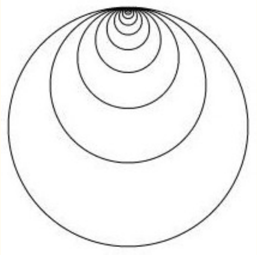
An essentially continuous strongly minimal theory

- $(\mathbb{R}, +)$ (with the appropriate metric) has a unique non-algebraic type over every parameter set (see picture).

Proposition (H.)

$\mathrm{Th}(\mathbb{R}, +)$ does not interpret an infinite discrete theory.

- N.B. The set $(-\infty, 0] \cup \{\ln n : 0 < n < \omega\}$ is definable in $(\mathbb{R}, +)$, but is neither compact nor co-pre-compact.



$S_1(\mathfrak{A})$ for a typical $\mathfrak{A} \succ \mathbb{R}$.

Approximately Strongly Minimal Pairs

Definition

(D, φ), with D a non-algebraic definable set and φ a formula, is an *approximately strongly minimal pair* if $\inf_{x \in D} \varphi(x) = 0$ and for every pair $F, G \subseteq D$ of disjoint zerosets and every $\varepsilon > 0$, at least one of $F \cap [\varphi \leq \varepsilon]$ and $G \cap [\varphi \leq \varepsilon]$ can be covered by finitely many open ε-balls in any model.

If (D, φ) is an approximately strongly minimal pair, then $D \cap [\varphi = 0]$ contains a unique non-algebraic type that is strongly minimal. We say that (D, φ) 'points to' p.

dic·tion·ar·ic

adjective

Of or pertaining to a dictionary.

Bringing strongly minimal imaginaries down to the prime model?

Partial result:

Proposition (H.)

If T is an inseparably categorical theory with a discrete strongly minimal imaginary then it has a strongly minimal imaginary over the prime model.