Separable and inseparable Gromov-Hausdorff categoricity in continuous logic

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Outline

1. Background
   - Continuous logic
   - Gromov-Hausdorff distance
2. Approximate categoricity
   - Separable approximate categoricity
   - Inseparable approximate categoricity
3. New Phenomena
   - Do ω-categorical, strictly ω_1-GH-categorical theories exist?
   - Elementary Gromov-Hausdorff distance

Syntax of continuous logic

- Continuous logic is a generalization of first-order logic. "Fuzzy logic on metric spaces."
- Connectives are arbitrary continuous functions from $[0,1]^n$ to $[0,1]$. Quantifiers are $\sup$ and $\inf$.
- A signature/language is mostly as it is in first-order logic (list of constant symbols and relation and function symbols with designated arities), except...
- A crucial new element is that each relation R or function f symbol has a fixed modulus of uniform continuity α_R or α_f that is *specified as part of the language.*
- This means that every formula is uniformly continuous and the modulus of uniform continuity can be computed *syntactically*.

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Semantics of continuous logic

- For a continuous language $\mathcal{L}$, a metric $\mathcal{L}$-structure, $\mathfrak{M}$, is a complete metric space of diameter at most 1 together with points, $[0, 1]$-valued predicates, and functions corresponding to the constant, relation, and function symbols of $\mathcal{L}$.
- The relations and functions need to obey the corresponding moduli of uniform continuity.
- For an $\mathcal{L}$-sentence φ, we say that $\mathfrak{M} \models \varphi$ if φ evaluates to 0 when computed in $\mathfrak{M}$. (Rationale: $d(x, y) = 0$ is the same thing as $x = y$.)

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Categoricity

Definition

A continuous first-order theory T is κ-categorical for cardinality κ if it only has one model of metric density character κ up to isomorphism.

Theorem (Ben Yaacov, Berenstein, Henson, Usvyatsov)

A countable theory T is ω-categorical iff every $\varnothing$-type is principal iff every $S_n(T)$ is metrically compact (think "finite").

Theorem (Ben Yaacov; Shelah, Usvyatsov)

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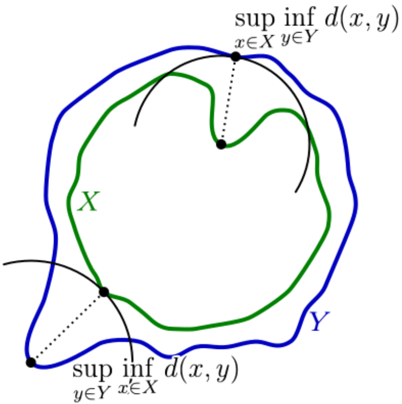
The Hausdorff metric



The Gromov-Hausdorff metric

Definition

For metric spaces X and Y,

$$d_{GH}(X, Y) = \inf\{d_H(\alpha(X), \beta(Y)) | \alpha : X \to Z, \beta : Y \to Z\},$$

for α, β isometric embeddings.

Logical aspects of Gromov-Hausdorff distance

- $d_{GH}(X, Y) = 0$ does not imply $X \cong Y$.
- $d_{GH}(X, Y) = 0$ does imply :
  - $X \equiv Y$
  - X and Y have the same density character and covering numbers (i.e. they're 'the same size').
  - For any non-principal ultrafilter $\mathcal{F}$ on ω, $X^\omega/\mathcal{F} \cong Y^\omega/\mathcal{F}$.
- d_{GH} can also be defined in terms of 'correlations' which can be seen as very strong back-and-forth strategies.
- Has natural generalization to Lipschitz languages (although language dependent), but also. . .

Theorem (H.)

Every countable theory is bi-interpretable with the theory of a pure metric space.

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Weak Ryll-Nardzewski characterization

Definition

A theory T is κ-GH-categorical if for any two $\mathfrak{M}, \mathfrak{N} \models T$ with metric density character κ, $d_{GH}(\mathfrak{M}, \mathfrak{N}) = 0$.

Example of ω-GH-categorical: 'Dense discrete pairs.'

Theorem (H., but essentially Ben Yaacov)

For a countable theory T:

- *If every $\varnothing$-type is 'GH-principal,' then T is ω-GH-categorical.*
- *A countable theory T is ω-GH-categorical if and only if every $\bar{a}$-type is 'weakly GH-principal' for every finite tuple $\bar{a}$.*

The first converse does fail.

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Morley's theorem?

- Example of an ω_1-GH-categorical theory: 'sin/cos fenceposts.'
- The 'hard' direction works:

Theorem (H.)

- *If a countable theory T is κ-GH-categorical for some $\kappa \geq \omega_1$, then every model $\mathfrak{M}$ with metric density character κ is 'GH-saturated.'*
- *If every $\mathfrak{M} \models T$ with metric density character κ is 'GH-saturated' for some $\kappa \geq \omega_1$, then the same is true for every $\lambda \geq \omega_1$.*

- The 'easy' direction (every κ sized model is GH-saturated $\Rightarrow$ T is κ-GH-categorical) is entirely unclear.
- Problem with 'accumulation of error' at ω_1.

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Which combinations are known to exist?

	ω_1-cat.	Strictly ω_1-GH-cat.	N/A
ω-cat.	Discrete set	?????	DLO
Strictly ω-GH-cat.	Pairs limiting to 0	Tagged sin/cos fenceposts	Dense discrete pairs
N/A	$\mathbb{Q}$-vector space	sin/cos fenceposts	ZFC

- In the very simple case that the metric is uniformly discrete, the missing square is provably impossible.
- Related discrete model theory question: Does there exist a sequence of countable languages $\mathcal{L}_0 \subseteq \mathcal{L}_1 \subseteq \cdots \subseteq \bigcup_{n<\omega} \mathcal{L}_n = \mathcal{L}$ and an $\mathcal{L}$-theory T such that T is ω-categorical, T is not ω_1-categorical, but for every $n < \omega$, $T \restriction \mathcal{L}_n$ is ω_1-categorical?

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$$d_{\preceq GH}(\mathfrak{M}, \mathfrak{N}) = \inf\{d_H(\alpha(\mathfrak{M}), \beta(\mathfrak{N})) | \alpha : \mathfrak{M} \preceq \mathfrak{C}, \beta : \mathfrak{N} \preceq \mathfrak{C}\},$$

for α, β elementary embeddings.

Again, $d_{\preceq GH}(\mathfrak{M}, \mathfrak{N}) = 0$ does not imply $\mathfrak{M} \cong \mathfrak{N}$, but...

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For any infinite κ, approximate κ-categoricity with regards to $d_{\preceq GH}$ implies κ-categoricity.

Counting models with regards to $d_{\preceq GH}$-density character of the space of models?

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