Definability and Categoricity in Continuous Logic

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Definability

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- Example: A structure M satisfies $\sup_{xyz} \max\{d(x,y), d(y,z)\} - d(x,z) \leq 0$ if and only if (M, d) is an ultrametric space (i.e. $\forall xyz(d(x,z) \leq \max\{d(x,y), d(y,z)\})$).

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- *Zeroset* of a formula is the set of all tuples where it evaluates to 0. (Also refers to corresponding set of types.)

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- Not every formula corresponds to a definable set!
- **Not closed under intersections!**

Type Spaces in Continuous Logic

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- $S_n(T)$ may fail to be zero-dimensional.
- Continuous function $S_n(T) \to \mathbb{R}$ correspond precisely to formulas with free variables among $\bar{x}$ (modulo T).
- Points (called *types*) are maximal consistent sets of real values for formulas.

The Metric on Type Space

- For types $p, q \in S_n(T)$,

$$d(p, q) = \inf\{d^M(\bar{a}, \bar{b}) : M \models p(\bar{a}), q(\bar{b})\}.$$

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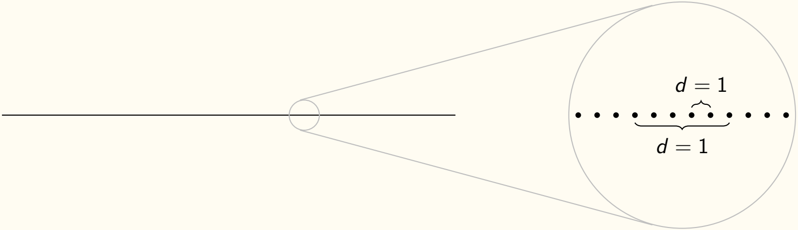
- Similar to the relationship between weak* and norm topologies on the unit ball of dual Banach spaces.
- A closed subset $D \subseteq S_{\bar{x}}(T)$ is definable if and only if $D^{<\varepsilon}$ is open for every $\varepsilon > 0$.

Definable Sets Can Be Poorly Behaved



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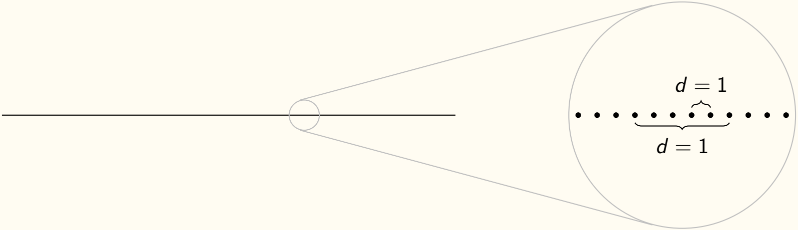

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There is also a type space homeomorphic to $[0, 1]$ with $d(x, y) = \max\{x, y\}$ for $x \neq y$. Has precisely 1 non-trivial definable set.

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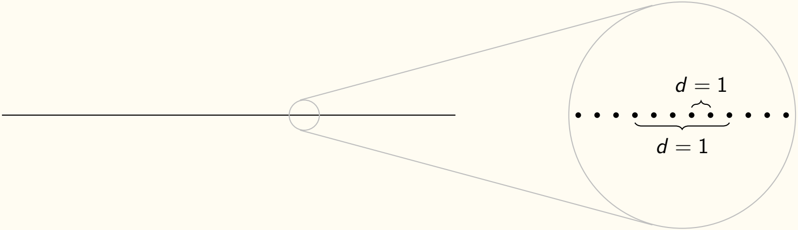

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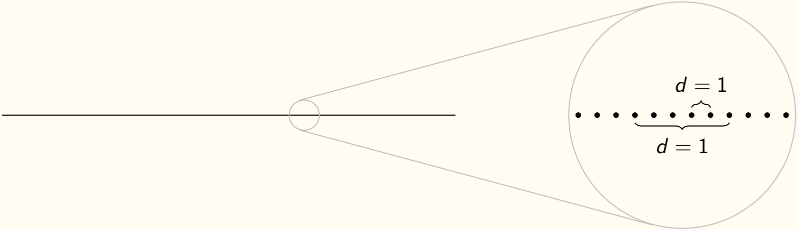

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Idea: Build a circuit.

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Recall that N is a neighborhood of p if $p \in \mathrm{int} N$.

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Examples: Discrete theories and randomizations of discrete theories.

Dictionaric Type Spaces II

Theorem (H.)

The following are equivalent:

1. $S_n(T)$ is dictionaric.
2. Definable sets separate disjoint closed subsets of $S_n(T)$.
3. For every disjoint closed $F, G \subseteq S_n(T)$, there is a definable set D such that either $F \subseteq D$ and $D \cap G = \varnothing$ or $G \subseteq D$ and $D \cap F = \varnothing$.
4. $S_n(T)$ has a network of definable sets (i.e. for every $p \in U \subseteq S_n(T)$, there is a definable set D such that $p \in D \subseteq U$).
5. For every $\varepsilon > 0$, $S_n(T)$ has a basis of open sets U satisfying $\mathrm{cl} U \subseteq U^{<\varepsilon}$.

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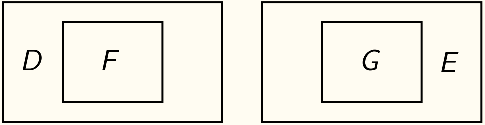

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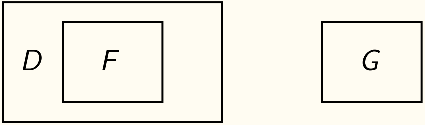

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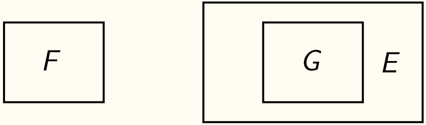

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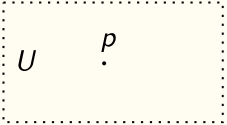

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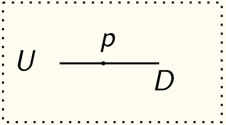

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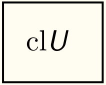

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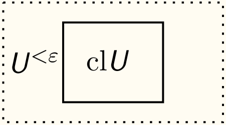

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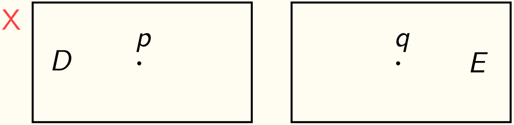
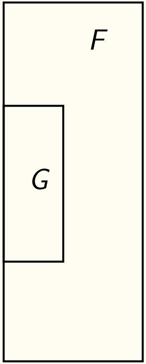


Nice Properties of Dictionaric Type Spaces I

Proposition (Extension)

If $S_n(T)$ is dictionaric and $G \subseteq F \subseteq S_n(T)$ are closed sets such that G is 'relatively definable in F' (for every $\varepsilon > 0$, $G \subseteq \mathrm{int}_F G^{<\varepsilon}$), then there is a definable set $D \subseteq S_n(T)$ such that $D \cap F = G$.



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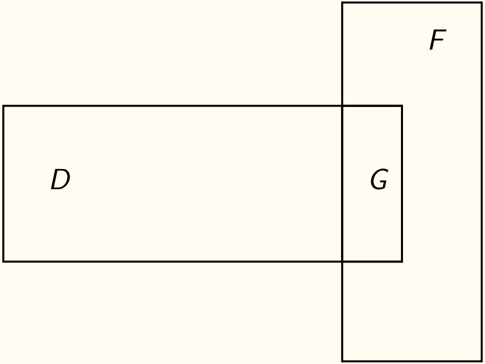


Nice Properties of Dictionaric Type Spaces II

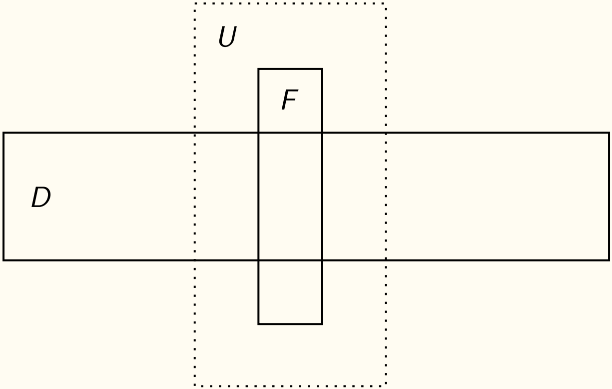
Proposition (Hereditariness to Definable Subsets)

If $S_n(T)$ is dictionaric and $D \subseteq S_n(T)$ is definable, then D is dictionaric as well.

Nice Properties of Dictionaric Type Spaces III

Proposition (Approximate Intersection)

If $S_n(T)$ is dictionaric, $D \subseteq S_n(T)$ is definable, and $F \subseteq U \subseteq S_n(T)$ are closed and open, respectively, then there is a definable set E such that $F \subseteq E \subseteq U$ and $D \cap E$ is definable.



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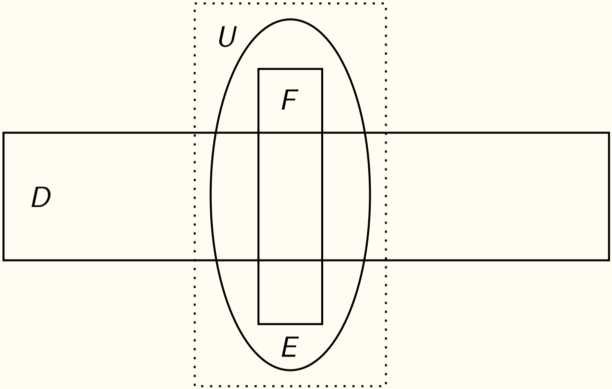

Nice Properties of Dictionaric Type Spaces III

Proposition (Approximate Intersection)

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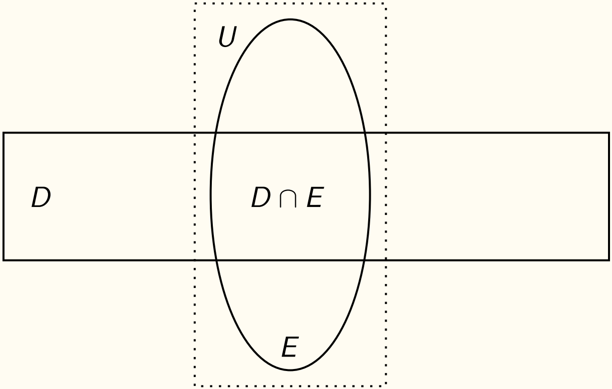

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Let X be a normal topological space. An *ordered separator* is an ordered pair of disjoint open sets. An ordered separator (U, V) is *strict* if $\mathrm{cl}U \cap \mathrm{cl}V = \varnothing$.

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Give a set of ordered separators P, the *separator game with payoff set P* is a game in which two players alternate playing strict separators (U_i, V_i) satisfying $U_i \supseteq U_{i-1}$ and $V_i \supseteq V_{i-1}$. Player II wins if and only if $(\bigcup U_i, \bigcup V_i) \in P$. A set of ordered separators is *generic* if Player II has a winning strategy in the separator game with payoff set P.

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I	U_0, V_0
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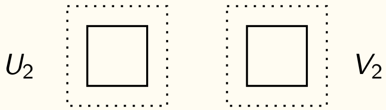

I	U_0, V_0		U_2, V_2
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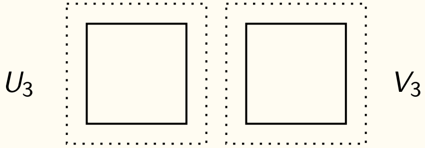

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II		U_1, V_1		U_3, V_3

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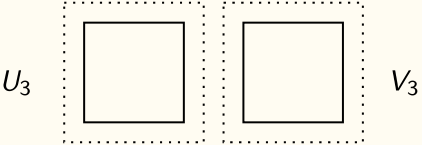

I	U_0, V_0		U_2, V_2	
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Example Application: A compact metric space has topological dimension $\leq n$ if and only if $\{(U, V) : \dim(X \setminus (U \cup V)) \leq n - 1\}$ is generic.

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- $D(x, y) \cap D(y, x)$ is definable (but not generic).

Categoricity

Uncountably Categorical Theories

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A theory is κ-*categorical* if it has a unique model of cardinality κ.

Theorem (Morley)

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These ingredients give you: A set with a good dimension theory (strongly minimal, from ω-stable) that 'controls' everything (no Vaughtian pairs).

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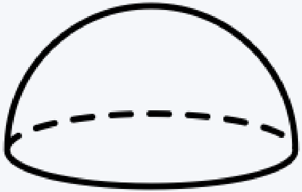
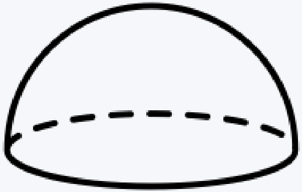


$S_1(H)$ for $H \models$ IHS.
(Not drawn topologically.)

Trouble with the Classical Picture

- The theory of (the unit ball of) an infinite dimensional Hilbert space, IHS, is inseparably categorical, but...
- ...does not have any strongly minimal types (see picture).
- IHS does not even interpret a strongly minimal theory.



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Has a unique non-algebraic type over any parameters. Such types are also called *strongly minimal*.

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The type space $S_1(\varnothing)$ of $\mathrm{Th}(A)$, topologically homeomorphic to $\omega + 1$. Limit type is strongly minimal but not contained in a $\varnothing$-definable strongly minimal set.

Approximately Strongly Minimal Pairs

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If T is a dictionaric theory with no Vaughtian pairs, then minimal sets are strongly minimal.

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When can we find strongly minimal types?

Two Axes of Difficulty

Continuous logic introduces two new difficulties:

- Lack of local compactness (of models).
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IHS has both of these issues, but can we tackle one of them at a time?

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Example: The theory of $(\mathbb{R}, +)$ with the metric $\min\{|x-y|, 1\}$, which is strongly minimal but does not interpret a discrete strongly minimal theory.

Ultrametrics and Totally Disconnected Type Spaces

Proposition (H.)

A theory T has totally disconnected type spaces iff it is dictionaric and has a $\varnothing$-definable ultrametric that is uniformly equivalent to the metric. Such theories are bi-interpretable with many-sorted discrete theories.

Not all ultrametric theories are dictionaric.

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Can We Improve That? I

Can we bring the assumption down to ‘no Vaughtian pairs’ rather than ‘no imaginary Vaughtian pairs’ for ultrametric theories?

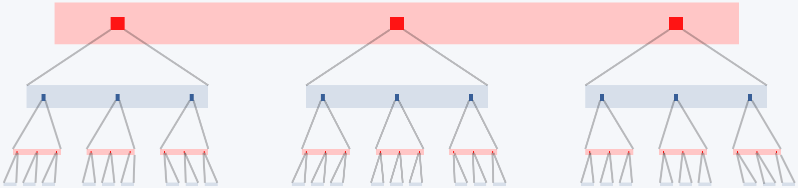
Can We Improve That? I

Can we bring the assumption down to ‘no Vaughtian pairs’ rather than ‘no imaginary Vaughtian pairs’ for ultrametric theories? No:

Theorem (H.)

There is an ω-stable ultrametric theory with no Vaughtian Pairs$^+$ which fails to be inseparably categorical.

Can We Imporve That? II



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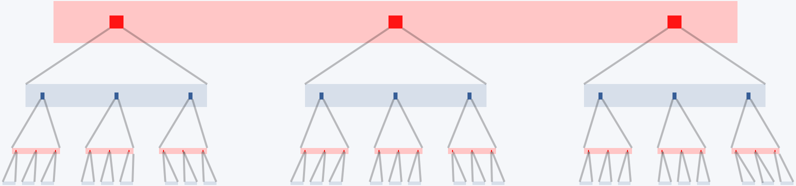

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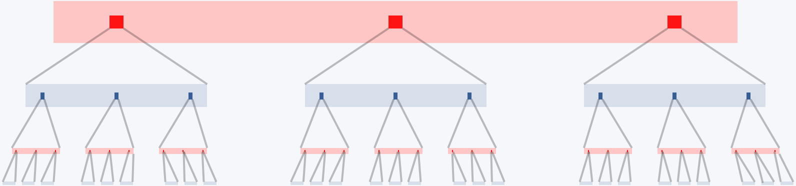

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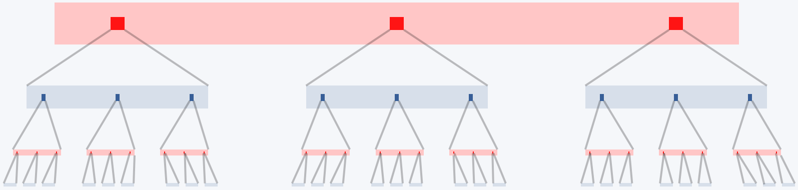
Can We Imporve That? II



Idea: Take two discrete affine spaces (i.e. vector spaces with 0 'forgotten'), V and W. Let $A = (V \times W)^{\omega}$ with an appropriate product metric. Give A the structure of an infinite wreath product: Add relations that give a bijection between any pair $\{\sigma\} \times V$ and $\{\tau\} \times V$ for any $\sigma, \tau \in (V \times W)^{<\omega}$ with even length if provided σ, τ, and an element of both of the copies of V as parameters (and likewise for W with σ and τ of odd length).

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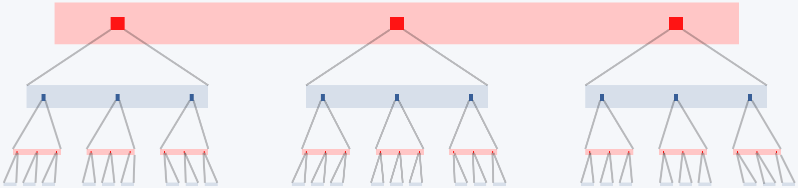

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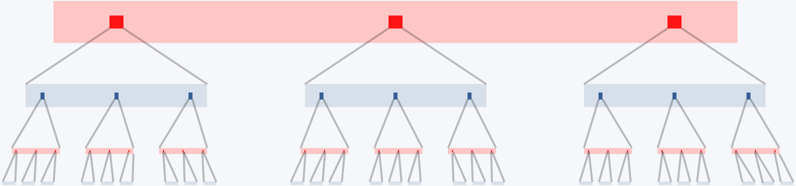

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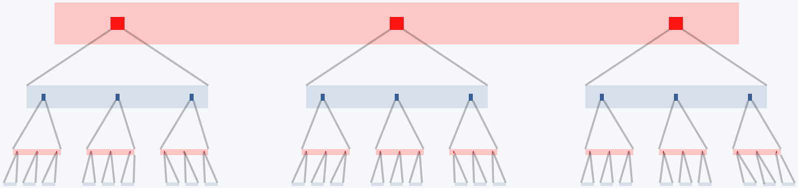

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Thank you