A Versatile Counterexample for Invariant Types and Keisler Measures outside NIP

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Joint work with Gabriel Conant and Kyle Gannon.

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Types and Measures

Invariant Types

Definition

Given a set of parameters A, a global type $p(x)$ is *A-invariant* if for any formula $\varphi(x, y)$ and any two tuples b, c with $b \equiv_A c$,

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Prototypical example (DLO):

Definable Types

Definition (Fiber Functions)

For each formula $\varphi(x, y)$, let $F_p^\varphi : S_y(A) \to \{0, 1\}$ be the function defined by $F_p^\varphi(q) = 1$ if $\varphi(x, b) \in p(x)$ for any $b \models q$.

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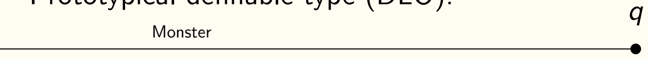

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- p, q definable $\Rightarrow$ $p \otimes q$ definable.
- p definable and q finitely satisfiable $\Rightarrow$ $p \otimes q = q \otimes p$.

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- An o-minimal theory has no non-trivial *dfs* types but does have non-trivial *dfs* measures.

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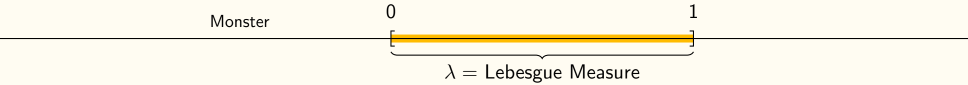

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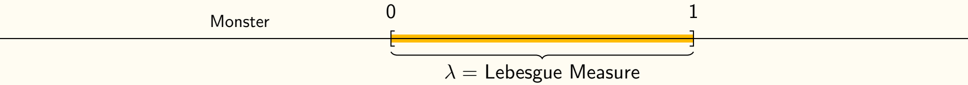

- There is also an intermediate property (which is non-trivial for types)...

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The type in the Henson graph is *fam* but not *fim*/generically stable (uses Erdös-Rogers).

Questions and Some Answers

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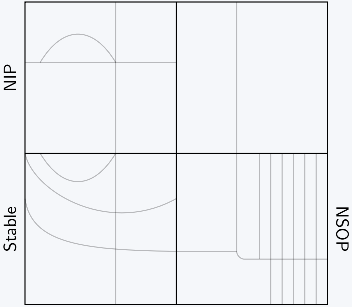
Theorems (Conant, Gannon, H.)

Over uncountable models of non-NIP theories, the Morley product of Borel definable measures may fail to be Borel definable and the Morley product of measures may fail to be associative (even when all products are Borel definable).

Half-Full of Half-Opens

dfs, ¬*fam* is Hard



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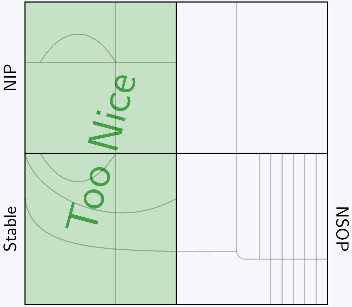

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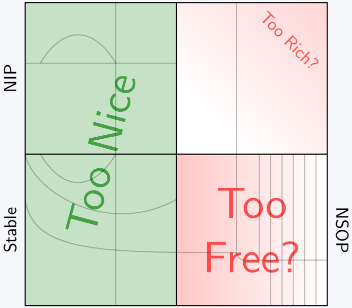

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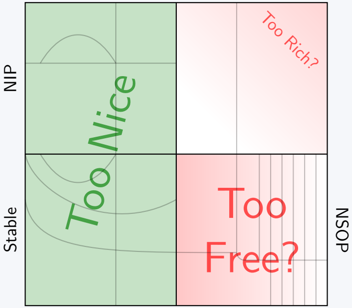

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Rules out theories that are too tame (NIP) *and* theories that are too rich (PA, ZFC).

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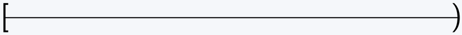

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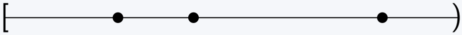

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$M_{1/2} = ([0, 1), \mathcal{H}_{1/2}, \in)$ gives a *local* example of a *dfs* type that is not *fam*: The $\in$-type $q(y)$ saying that every element of the $[0, 1)$-sort is in y is *dfs*.

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- $M_{1/2}$ interprets a Boolean algebra ($\mathcal{H}$).

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Proposition (Conant, Gannon, H.)

Any expansion of a Boolean algebra has no non-trivial *dfs* types.

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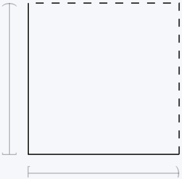

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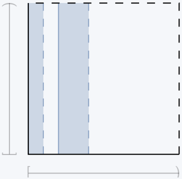

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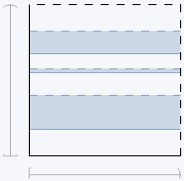

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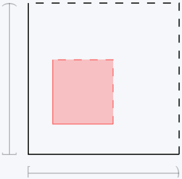

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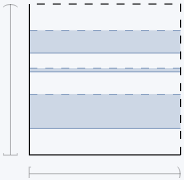

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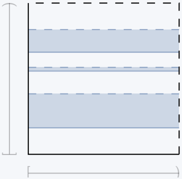

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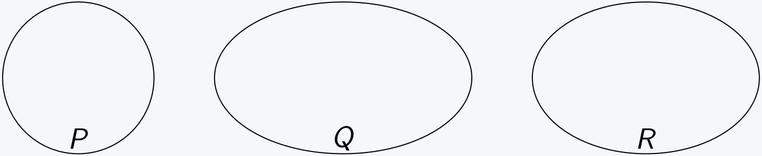

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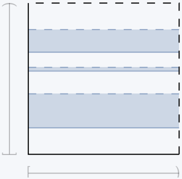

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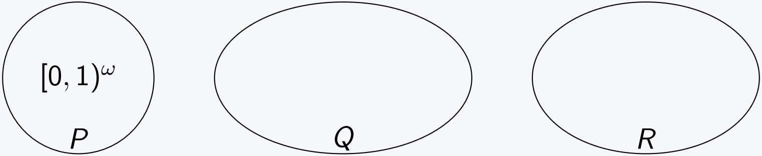

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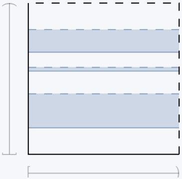

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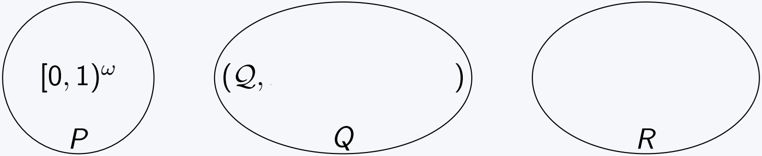

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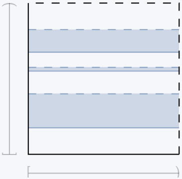

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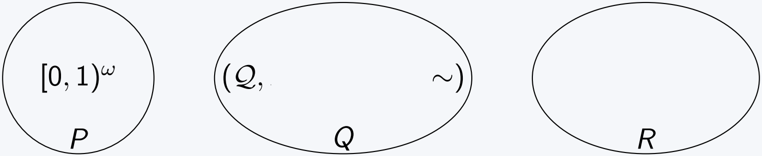

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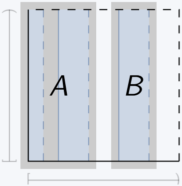

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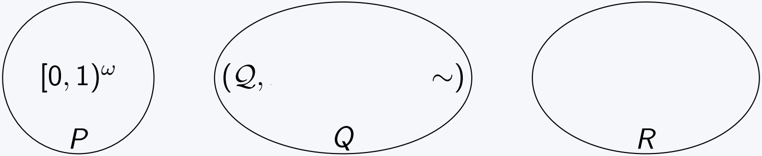

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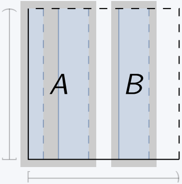

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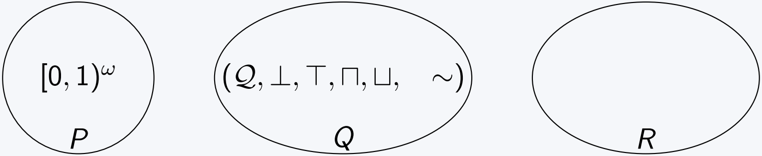

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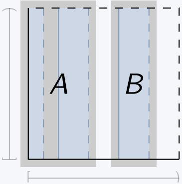

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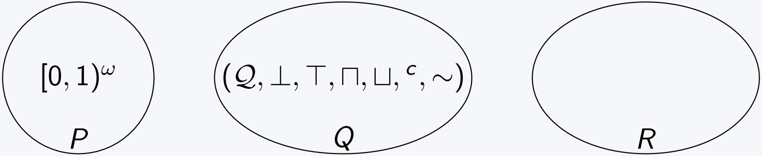

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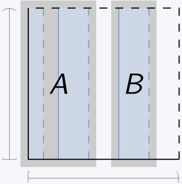

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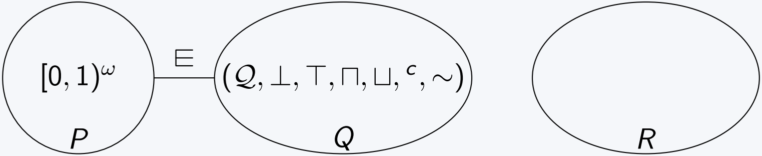

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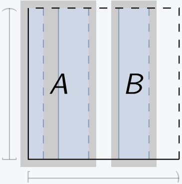

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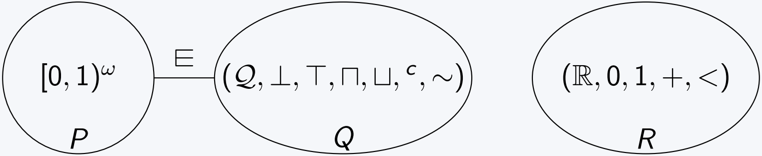

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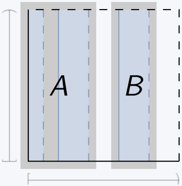

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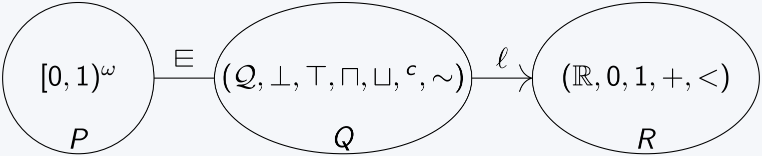

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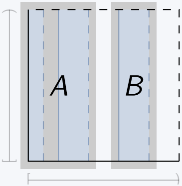

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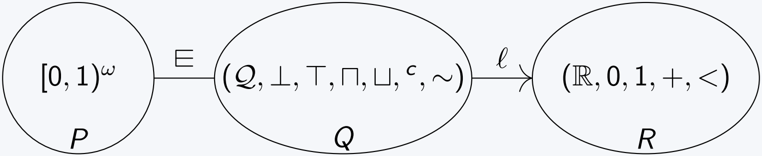

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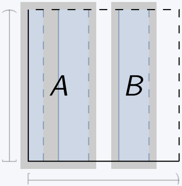

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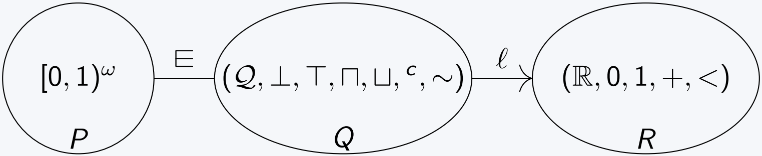

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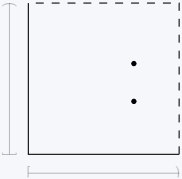

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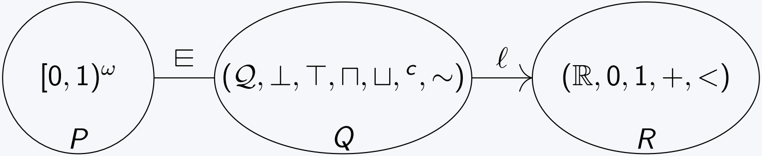

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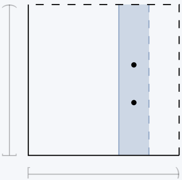

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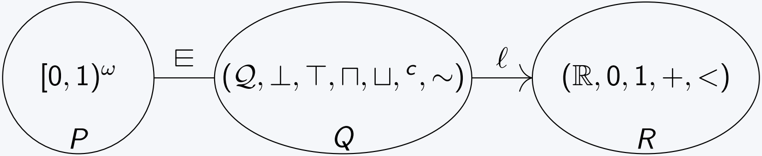

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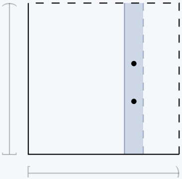

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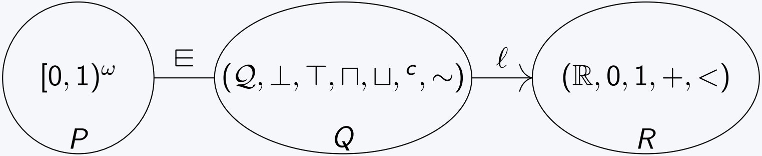

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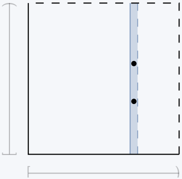

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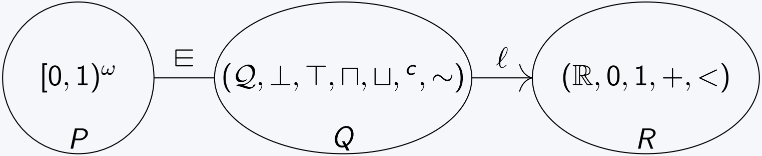

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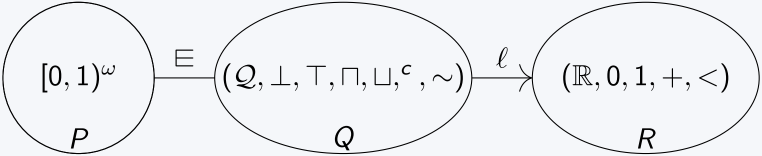

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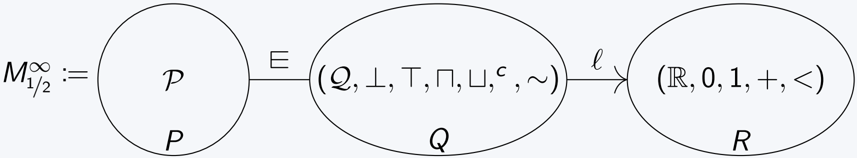

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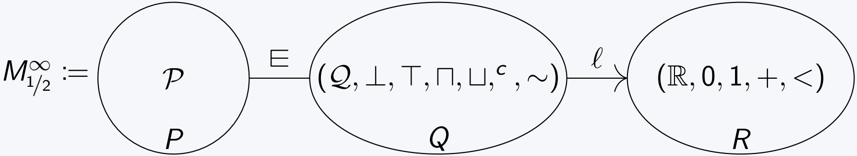

Theorem (Conant, Gannon, H.)

$T^\infty_{1/2} := \mathrm{Th}(M^\infty_{1/2})$ has quantifier elimination.

QE Proof (Sketch)

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Definition

Let $q_{1/2}(y)$ be the type in the Q sort axiomatized by

- $a \sqsubseteq y$ for all $a \in P(\mathcal{U})$,
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- Complete, definable type by QE.
- Finitely satisfiable in $M^{\infty}_{1/2}$ (therefore consistent).

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- $\{b_i\}_{i<n}$ fails to approximate the behavior of $\varphi(a,y)$. □

But Wait, There's More

New Example of *fam*, $\neg$Generically Stable

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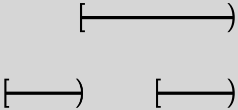

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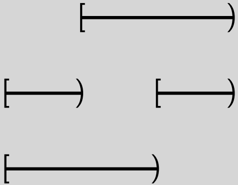

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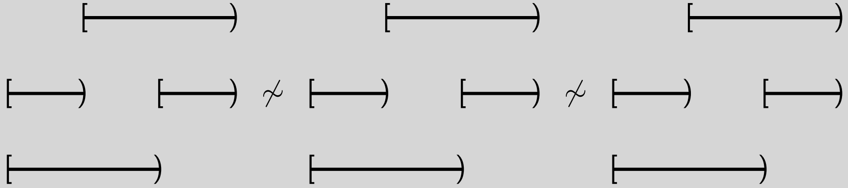

$\square$

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For example, if b, c, d are pairwise $\sim$-inequivalent, then

$$\mu(x \sqsubseteq b \wedge x \sqsubseteq c \wedge x \sqsubseteq d) = \mathrm{st}(\ell(b)\ell(c)\ell(d)),$$

where st is the standard part map.

Failure of Commutativity

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In particular, there are a *dfs* type and a definable measure that do not commute.

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Thank you