A Versatile Counterexample for Invariant Types and Keisler Measures outside NIP

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Joint work with Gabriel Conant and Kyle Gannon.

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Types and Measures

Invariant Types

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Given a set of parameters A, a global type $p(x)$ is *A-invariant* if for any formula $\varphi(x, y)$ and any two tuples b, c with $b \equiv_A c$,

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Prototypical example (DLO):

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For each formula $\varphi(x, y)$, let $F_p^{\varphi} : S_y(A) \to \{0, 1\}$ be the function defined by $F_p^{\varphi}(q) = 1$ if $\varphi(x, b) \in p(x)$ for any $b \models q$.

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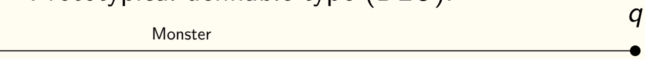

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- p definable and q finitely satisfiable $\Rightarrow$ $p \otimes q = q \otimes p$.

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- An o-minimal theory has no non-trivial *dfs* types but does have non-trivial *dfs* measures.

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Such measures do not always exist (but they do if μ is Borel definable or if ν is a type).

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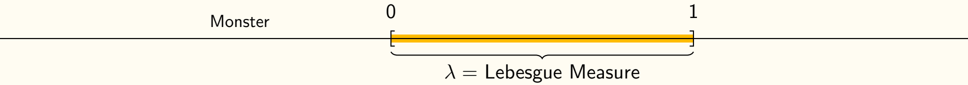

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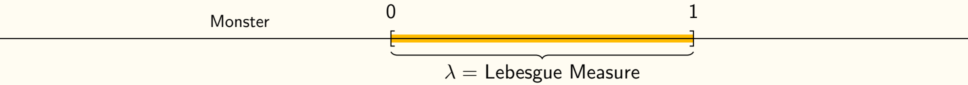

- There is also an intermediate property (which is non-trivial for types)...

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Nice Measures II

Definition

A measure $\mu(x)$ is *fam* (finitely approximable measure) if there is some small model M such that for any formula $\varphi(x, y)$ and any $\varepsilon > 0$, there are $\bar{a} \in (M^x)^n$ such that

$$\left| \mu(\varphi(x, b)) - \frac{1}{n} \sum_{i<n} \varphi(a_i, b) \right| < \varepsilon$$

for all b in the monster.

We say that p is *fam* if δ_p is *fam*. In general,

$$\textit{dfs} \Leftarrow \textit{fam} \Leftarrow \textit{fim}.$$

In NIP theories, *dfs* measures are always *fim*. The type in the Henson graph is *fam* but not *fim*/generically stable (uses Erdös-Rogers).

Questions and Some Answers

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In fact, there are Borel definable types $p(x)$ and $q(y)$ and a Borel definable measure $\lambda(z)$ (in the theory of a random ternary relation) such that $(p \otimes q) \otimes \lambda \neq p \otimes (q \otimes \lambda)$.

Half-Full of Half-Opens

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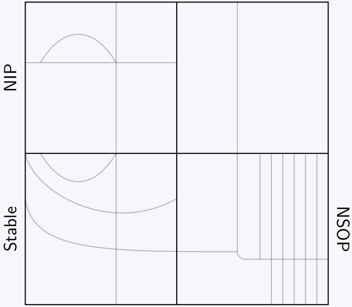

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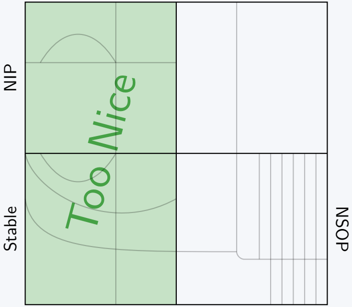

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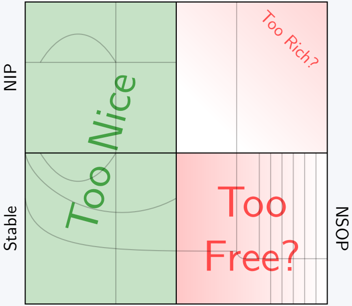

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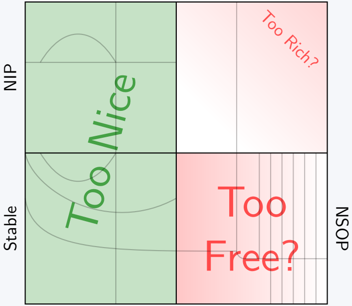

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Together these rule out theories that are too tame (NIP) *and* theories that are too rich (PA, ZFC).

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The two-sorted structure $M_{1/2} = ([0,1), \mathcal{H}_{1/2}, \in)$ gives a *local* example of a *dfs* type that is not *fam*: The $\in$-type $q(y)$ saying that every element of the $[0,1)$-sort is in y is *dfs*.

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- $M_{1/2}$ interprets a Boolean algebra ($\mathcal{H}$).

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Proposition (Conant, Gannon, H.)

Any expansion of a Boolean algebra has no non-trivial *dfs* types.

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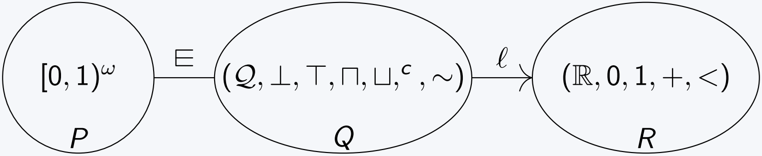

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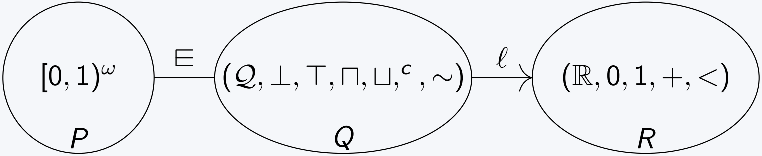

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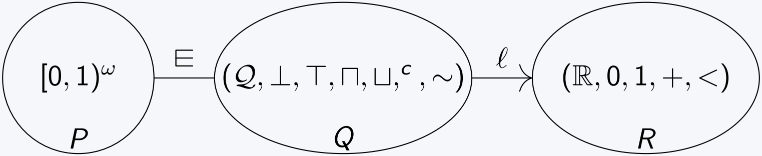

The point of all this structure is to get QE, but this structure doesn't actually have QE: $(\forall z \sim b) x \sqsubseteq z \leftrightarrow y \sqsubseteq z$.

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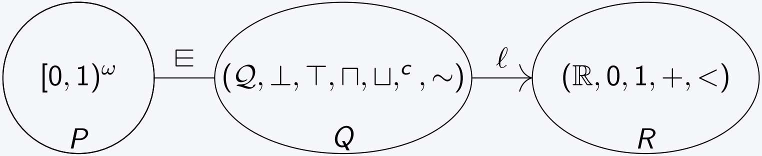

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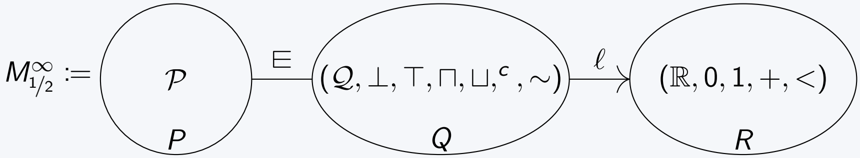

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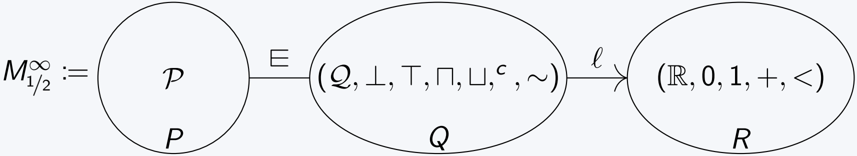

Theorem (Conant, Gannon, H.)

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- Finitely satisfiable in $M^{\infty}_{1/2}$ (therefore consistent).

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- $\{b_i\}_{i<n}$ fails to approximate the behavior of $\varphi(a,y)$. □

But Wait, There's More

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For any n, fix let $\{b_i\}_{i<n^2}$ be an enumeration of elements of $\mathcal{Q}$ of the form (k, A) with $k < n$ and $A = [0,1) \setminus [\frac{j}{n}, \frac{j+1}{n})$ for some $j < n$.

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By QE, this is enough to establish that q_{PQ} is *fam* over $M^{\infty}_{1/2}|_{PQ}$. □

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Think of μ as randomly picking an element of P with each 'coordinate' distributed independently according to ℓ.
For example, if b, c, d are pairwise $\sim$-inequivalent, then

$$\mu(x \sqsubseteq b \wedge x \sqsubseteq c \wedge x \sqsubseteq d) = \mathrm{st}(\ell(b)\ell(c)\ell(d)),$$

where st is the standard part map.

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In particular, there are a *dfs* type and a definable measure that do not commute.

Some Remaining Questions

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- Do any two *dfs* measures commute?

Thank you